Artificial Intelligence 336 (2024) 104203

Contents lists available at ScienceDirect

Artificial Intelligence

journal homepage: www.elsevier.com/locate/artint

Bisimulation between base argumentation and premise-conclusion argumentation ☆

Jinsheng Chen [a,*], Beishui Liao [b,d], Leendert van der Torre [c,b]

[a] Department of Philosophy (Zhuhai), Sun Yat-sen University, China
[b] School of Philosophy, Zhejiang University, China
[c] Department of Computer Science, University of Luxembourg, Luxembourg
[d] The State Key Lab of Brain-Machine Intelligence, China

ARTICLE INFO

Keywords:
Abstract argumentation
Structured argumentation
Deductive argumentation

ABSTRACT

The structured argumentation system that represents arguments by premise-conclusion pairs is called *premise-conclusion argumentation* (PA) and the one that represents arguments by their premises is called *base argumentation* (BA). To assess whether BA and PA have the same ability in argument evaluation by extensional semantics, this paper defines the notion of *extensional equivalence* between BA and PA. It also defines the notion of *bisimulation* between BA and PA and shows that bisimulation implies extensional equivalence. To illustrate how base argumentation, bisimulation and extensional equivalence can contribute to the study of PA, we prove some new results about PA by investigating the extensional properties of a base argumentation framework and exporting them to two premise-conclusion argumentation frameworks via bisimulation and extensional equivalence. We show that there are essentially three kinds of extensions in these frameworks and that the extensions in the two premise-conclusion argumentation frameworks are identical.

1. Introduction

Formalizing arguments through logic enables the direct utilization of tools developed in logic, including formal languages for knowledge representation and various models and calculi for deciding logical consequences. Given an argument consisting of declarative sentences with one conclusion, after determining a logic, generally the argument can be represented in two different levels of abstraction:

(1) Representing the argument as a sequence of formulas. In addition to translating each declarative sentence into a formula in the formal language, the representation also indicates the logical relation between the formulas (e.g., it may say that ψ is obtained from $\varphi \to \psi$ and φ by *modus ponens*). In this case, an argument is a sequence of formulas, in which each formula is either a logical axiom, an assumption, or can be derived from the preceding formula(s) by an inference rule.

(2) Representing the argument as a premise-conclusion pair. After determining the premises and the conclusion of the argument through manual analysis, they are translated into formulas. Collect the premises into a set Γ and denote the conclusion by a formula φ.

☆ This paper is an extension of work originally presented in [13].

* Corresponding author.
E-mail address: jinsheng.chen@foxmail.com (J. Chen).

https://doi.org/10.1016/j.artint.2024.104203
Received 17 November 2023; Received in revised form 8 August 2024; Accepted 10 August 2024

Available online 20 August 2024

The argument is then represented as a premise-conclusion pair (Γ, φ). In the representation, the process of logical reasoning is hidden and a derivation from Γ to φ is assumed to be available.

A less common way of representing arguments is to represent the argument by its premises. In this way of representation, the process of logical reasoning as well as the conclusion is hidden, and as a result, a formal argument can represent more than one argument. This is a more compact way of knowledge representation.

Closely related to this way of representation is a hypothesis called *logical omniscience hypothesis*. It assumes that an intelligent agent has perfect logical ability and knows all the logical consequences after knowing the premises. This hypothesis is widely assumed in epistemic logic, but some criticize it as irrational [30]. After all, if it were true, it would not cost mathematicians hundreds of years to prove Fermat's last theorem. Due to the exceptional computing power of computers as well as the availability for algorithms to decide certain (fragments of) logics, this paper assumes that the logical omniscience hypothesis and representing an argument by its premises are reasonable in formal representation of everyday reasoning.

For example, consider the following arguments[1]:

> Argument A
>
> I stop existing without my body.
>
> If I stop existing without my body, then I am my body.
>
> Therefore, I am my body.
>
> Argument B
>
> I can imagine existing without my body.
>
> If I can imagine existing without my body, then I am not my body.
>
> Therefore, I am not my body.

For Argument A, we use φ to represent the first premise, ψ the conclusion, and $\varphi \to \psi$ the second premise. For Argument B, we use ξ to represent the first premise, $\neg\psi$ the conclusion, and $\xi \to \neg\psi$ the second premise. Then Argument A can be formally represented in three different ways, each corresponding to a way of argument representations mentioned above:

- 1. φ (premise); 2. $\varphi \to \psi$ (premise); 3. ψ (1, 2, modus ponens);
- $(\{\varphi, \varphi \to \psi\}, \psi)$;
- $\{\varphi, \varphi \to \psi\}$.

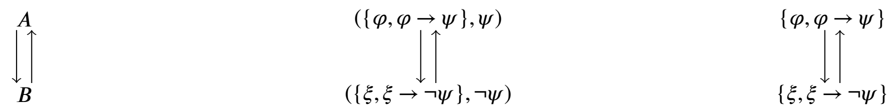
Similarly, argument B can be formally represented as follows:

- 1. ξ (premise); 2. $\xi \to \neg\psi$ (premise); 3. $\neg\psi$ (1, 2, modus ponens);
- $(\{\xi, \xi \to \neg\psi\}, \neg\psi)$;
- $\{\xi, \xi \to \psi\}$.

Arguments A and B are in conflict. Abstract argumentation [15] is a model for conflicting arguments, which views argumentation as a framework consisting of a set of arguments and a binary attack relation on arguments. A set of acceptable arguments is called an *extension*. An *extensional semantics* provides criteria for selecting extensions from the framework through the attack relation.

Structured argumentation [23,24,18,19,29,14,7,8] also views argumentation as a framework, but the arguments and attack relations are not longer atomic objects, and are defined in terms of logical and/or defeasible inference rules. Structured argumentation uses the same extensional semantics as abstract argumentation to decide which of the arguments are acceptable. Given a deductive logic, the structured argumentation system that represents arguments by their premises is called *base argumentation* (BA) [13] and the one that represents arguments by premise-conclusion pairs is called *premise-conclusion argumentation* (PA) [7,8].[2]

Arguments A and B may be graphically represented in abstract argumentation, PA and BA as follows respectively, where arrows represent attacks:



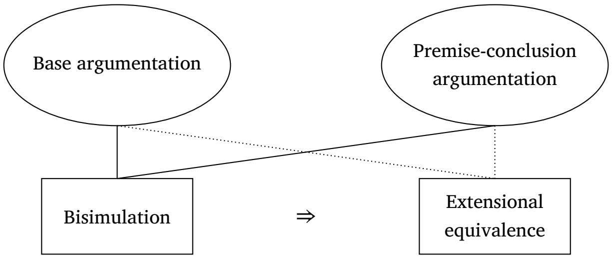
The question of interest in this paper is whether BA and PA have the same ability in argument evaluation via extensional semantics. With its obvious simplicity, if BA has the same ability in argument evaluation as PA, then BA is desirable as an argumentation system.

[1] Argument B is adapted from [27].

[2] Premise-conclusion argumentation is called *deductive argumentation* in the literature. Since base argumentation is also defined in terms of deductive logic, it is named in terms of the form of arguments in this paper to distinguish it from base argumentation.

To assess whether BA and PA have the same ability in argument evaluation, we define the notion of *extensional equivalence* between a *base argumentation framework* (BAF) and a *premise-conclusion argumentation framework* (PAF). Extensional equivalence is defined in terms of an operation that generates premise-conclusion arguments from premises and an operation that extracts premises from premise-conclusion arguments. The operations reflect the logical omniscience hypothesis mentioned above. Extensional equivalence requires the two operations to be bijections between the extensions of the BAF and the extensions of the PAF.

Moreover, we define the notion of *bisimulation between a BAF and a PAF*, which is similar to the notion of *bisimulation in transition systems* that describes the behavioral equivalences between transition systems (see Section 2.3.4 in [21]) and the notion of *bisimulation in modal logic* that preserves satisfiability between models (see Section 2.2 in [9]). We prove that bisimulation implies extensional equivalence. This is depicted graphically as follows:



To illustrate how base argumentation, bisimulation and extensional equivalence can contribute to the study of premise-conclusion argumentation, we conduct an in-depth study of the extensional properties of a BAF and export them to two PAFs via bisimulation and extensional equivalence. We show that there are essentially three kinds of extensions in these frameworks: complete, grounded and preferred extensions. We also show that the extensions in the two PAFs are identical.

This paper contributes to structured argumentation. Since BA and PA are extensionally equivalent under certain conditions, BA may be preferred in some cases because of its obvious simplicity. In addition, we consider seven kinds of extensional semantics in this paper and show that for a BAF and two PAFs, there are essentially three kinds of extensions. This contributes to the question of whether the various extensional semantics are essentially different. For a detailed discussion, see Section 9.

The rest of the paper is structured as follows. Section 2 reviews abstract argumentation, abstract logic and premise-conclusion argumentation. Section 3 defines base argumentation. Section 4 defines the notion of extensional equivalence. Section 5 defines the notion of bisimulation. Section 6 shows that bisimulation implies extensional equivalence. Section 7 investigates the extensional properties of a BAF. Section 8 exports the properties of the BAF in Section 7 to two PAFs via bisimulation and extensional equivalence. Section 9 discusses related works and concludes the paper.

2. Preliminaries

2.1. Abstract argumentation

Central to the theory of abstract argumentation is *abstract argumentation frameworks* [15], which are essentially directed graphs in which the arguments are represented by nodes and the attack relations are represented by arrows.

Definition 1. An *abstract argumentation framework* $\mathcal{F}$ is a pair $(\mathcal{A}, \mathcal{R})$ where $\mathcal{A}$ is a set of *arguments* and $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$. $\mathcal{R}$ is called the *attack relation* of $\mathcal{F}$. If $(A, B) \in \mathcal{R}$, we say that A *attacks* B.

Given an abstract argumentation framework $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, a set of arguments $S \subseteq \mathcal{A}$ *attacks* argument $A \in \mathcal{A}$ if there is an argument $A' \in S$ such that A' attacks A. We say that S *defends* A if for each argument $B \in \mathcal{A}$, if B attacks A, then S attacks B. For a set S of arguments, let $S^+ := \{A \mid S \text{ attacks } A\}$.

Various extensional semantics are considered in the literature to decide which arguments in an abstract argumentation framework are acceptable (see e.g., [15,31] [10,16,11]). In this paper, we consider the following semantics:

Definition 2. Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an argumentation framework and $S \subseteq \mathcal{A}$. S is *conflict-free* if there is no argument $A \in S$ such that S attacks A and is *admissible* if S is conflict-free and S defends each of its elements. Then we say that:

- S is a *complete* extension if S is admissible and contains each argument it defends.
- S is a *stable* extension if S is conflict-free and attacks each argument which does not belong to S.
- S is a *grounded* extension if S is the least complete extension.
- S is a *preferred* extension if S is a maximal complete extension.[3]

[3] In Dung [15], a preferred extension is defined to be a maximal admissible extension. These two definitions are equivalent.

– S is a *semi-stable* extension if S is a complete extension and $S \cup S^+$ is maximal among complete extensions, i.e., there exists no complete extension S' such that $S \cup S^+ \subset S' \cup S'^+$.
– S is an *ideal* extension if S is a maximal complete extension that is contained in each preferred extension.
– S is an *eager* extension if S is a maximal complete extension that is contained in each semi-stable extension.

In the above statements, maximality and minimality are considered with respect to set inclusion.

2.2. Abstract logic

Logical instantiations of abstract argumentation need a formal logic for representing arguments and defining attack relations. Various formal logics capturing different patterns of reasoning are studied in the literature. To make the results in this paper more general, *abstract logics* [28] are used as the underlying logic. An abstract logic is a pair of a formal language and a consequence relation on this language satisfying certain important logical properties. Many well-known formal logics like propositional logic, intuitionistic logic, epistemic logics and deontic logics are abstract logics defined below.

Abstract logics are defined in terms of *abstract consequence relations*.

Definition 3. Let $\mathcal{L}$ be a denumerable logical language. We use φ, φ, ψ for formulas in $\mathcal{L}$, and Γ, Δ for multisets of formulas in $\mathcal{L}$. A relation $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ is an *abstract consequence relation* if it satisfies the following conditions: for any $\Gamma, \Delta \subseteq \mathcal{L}$ and $\varphi, \psi \in \mathcal{L}$:

1. (Reflexivity) If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.
2. (Monotonicity) If $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \varphi$.
3. (Transitivity) If $\Gamma \vdash \varphi$ and $\{\varphi\} \cup \Gamma' \vdash \psi$, then $\Gamma \cup \Gamma' \vdash \psi$.

An *abstract logic* is a pair $(\mathcal{L}, \vdash)$ where $\vdash$ is an abstract consequence relation on $\mathcal{L}$.

If $\Gamma = \{\varphi_1, \dots, \varphi_n\}$ we often write $\varphi_1, \dots, \varphi_n \vdash \psi$ for $\{\varphi_1, \dots, \varphi_n\} \vdash \psi$. If Γ contains just one formula, e.g., $\Gamma = \{\varphi\}$, we write $\varphi \vdash \psi$ for $\{\varphi\} \vdash \psi$.

We assume that $(\mathcal{L}, \vdash)$ contains the following connectives and satisfies the corresponding properties:

– Falsity $\bot$: $\bot \vdash \varphi$ for any formula φ.
– Conjunction $\wedge$: (1) $\varphi, \psi, \Gamma \vdash \xi$ iff $\varphi \wedge \psi, \Gamma \vdash \xi$; (2) $\Gamma \vdash \varphi \wedge \psi$ iff $\Gamma \vdash \varphi$ and $\Gamma \vdash \psi$.
– Negation $\neg$: (1) $\varphi, \Gamma \vdash \bot$ iff $\Gamma \vdash \neg\varphi$; (2) $\Gamma \vdash \varphi$ iff $\neg\varphi, \Gamma \vdash \bot$.

It follows from the property of negation that $\varphi, \Gamma \vdash \psi$ iff $\neg\psi, \Gamma \vdash \neg\varphi$.

We use the following abbreviations: $\varphi \to \psi$ for $\neg(\varphi \wedge \neg\psi)$, $\varphi \leftrightarrow \psi$ for $(\varphi \to \psi) \wedge (\psi \to \varphi)$. For a finite set $\Gamma = \{\varphi_1, \dots, \varphi_n\}$, let $\bigwedge \Gamma := \varphi_1 \wedge \dots \wedge \varphi_n$.

Let $Cn_{\vdash}(\Gamma)$ be the set of logical consequences of Γ, i.e., $Cn_{\vdash}(\Gamma) := \{\varphi \mid \Gamma \vdash \varphi\}$.

A set Γ of $\mathcal{L}$-formulas is called $\vdash$-*inconsistent* if $\Gamma \vdash \bot$. Γ is called *consistent* if it is not inconsistent. When the consequence relation is clear from the context, we omit $\vdash$ and simply say that Γ is *consistent* or *inconsistent*.

Next we show that if a set of formulas implies a formula and its negation, then it is inconsistent.

Proposition 1. *For an abstract logic $(\mathcal{L}, \vdash)$ and a set Γ of formulas, if $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$ for some formula φ, then Γ is inconsistent.*

Proof. By the property of conjunction, $\Gamma \vdash \varphi \wedge \neg\varphi$. By the reflexivity of $\vdash$, $\varphi \vdash \varphi$. By the property of $\neg$, $\varphi, \neg\varphi \vdash \bot$. By the property of conjunction, $\varphi \wedge \neg\varphi \vdash \bot$. By the transitivity of $\vdash$, $\Gamma \vdash \bot$. □

2.3. Premise-conclusion argumentation

This subsection introduces the premise-conclusion argumentation defined in the literature (see e.g., [20]). It is a logical instantiation of abstract argumentation.

In premise-conclusion argumentation, it is assumed as given an abstract logic $(\mathcal{L}, \vdash)$, which provides the formal language to formalize relational data and the logical mechanism for automated reasoning.

The formalized data form a *knowledge base*. It is a subset of our logical language $\mathcal{L}$ and possibly inconsistent.

Definition 4. For an abstract logic $(\mathcal{L}, \vdash)$, a *knowledge base* Σ is a subset of $\mathcal{L}$.

Then we define the notion of *premise-conclusion arguments*. There are three requirements for premise-conclusion arguments: consistency of premises, logical entailment of the conclusion from the premises, and the absence of a proper subset of premises that can yield the same conclusion.

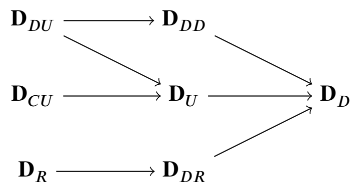


Fig. 1. Containment between attack relations on premise-conclusion arguments. An arrow from D_1 to D_2 indicates that $D_1 \subseteq D_2$.

Definition 5. Given an abstract logic $(\mathcal{L}, \vdash)$ and a knowledge base Σ, a *premise-conclusion argument* is a pair (Γ, φ) such that

1. Γ is a finite consistent subset of Σ.
2. $\Gamma \vdash \varphi$.
3. There is no $\Gamma' \subset \Gamma$ such that $\Gamma' \vdash \varphi$.

We use capital letters $A, B, \ldots$ for premise-conclusion arguments. For $A = (\Gamma, \varphi)$, let $\mathrm{S}(A) = \Gamma$ and call it the *premise set* of A, and let $\mathrm{C}(A) = \varphi$ and call it the *conclusion* of A. We say that (Γ, φ) is a *sub-argument* of (Δ, ψ) if $\Gamma \subseteq \Delta$.

Example 1. Let $(\mathcal{L}, \vdash)$ be propositional logic and knowledge base $\Sigma := \{p, \neg p, q\}$. $(\{p\}, p)$ is a premise-conclusion argument; $(\{p, q\}, p)$ is not a premise-conclusion argument, because $\{p\} \subset \{p, q\}$ and $(\{p\}, p)$ is a premise-conclusion argument; $\{p, \neg p\} \vdash p$ is not a premise-conclusion argument because its premise set is inconsistent.

Lemma 1. *[Existence of premise-conclusion arguments] Let $(\mathcal{L}, \vdash)$ be an abstract logic and Σ a knowledge base. If $\Gamma \subseteq \Sigma$ is consistent and $\Gamma \vdash \varphi$, then there exists a minimal subset Γ' of Γ such that $\Gamma' \vdash \varphi$ and (Γ', φ) is a premise-conclusion argument.*

Proof. Let $\varphi_1, \varphi_2, \ldots, \varphi_m$ be an enumeration of formulas in Γ. Denote by $\Gamma_i (1 \leq i \leq m)$ the set of formulas obtained from Γ by removing φ_i. If there is no Γ_i such that $\Gamma_i \vdash \varphi$, then (Γ, φ) is the required premise-conclusion argument. Otherwise, choose an arbitrary Γ_i such that $\Gamma_i \vdash \varphi$. Then we obtain a proper subset of Γ which deduces φ. Repeat the above process until we obtain a set Γ' such that for any $\psi \in \Gamma'$, $\Gamma' \setminus \{\psi\} \nvdash \varphi$. Since $\Gamma' \subseteq \Gamma$, by the monotonicity of $\vdash$, Γ' is consistent. By construction, Γ' is minimal. Therefore, (Γ', φ) is a premise-conclusion argument. $\square$

Next we define attack relations on premise-conclusion arguments. Let $Ar_p(\Sigma)$ be the set of premise-conclusion arguments on Σ. The attack relation on $Ar_p(\Sigma)$ is represented by a function $\mathbf{D} : Ar_p(\Sigma) \times Ar_p(\Sigma) \to \{0, 1\}$. If $\mathbf{D}(A, B) = 1$, we say that A $\mathbf{D}$*-attacks* B. The following attack relations are considered in [20]. We use the following acronyms: defeater (D), direct defeater (DD), undercut (U), direct undercut (DU), canonical undercut (CU), rebuttal (R), defeating rebuttal (DR).

Definition 6. Given premise-conclusion arguments $(\Gamma, \varphi), (\Delta, \psi)$,

- (Γ, φ) $\mathbf{D}_D$*-attacks* (Δ, ψ), if $\varphi \vdash \neg \bigwedge \Delta$;
- (Γ, φ) $\mathbf{D}_{DD}$*-attacks* (Δ, ψ), if there is $\xi \in \Delta$ such that $\varphi \vdash \neg \xi$;
- (Γ, φ) $\mathbf{D}_U$*-attacks* (Δ, ψ), if there is $\Delta' \subseteq \Delta$ such that $\varphi \equiv \neg \bigwedge \Delta'$;
- (Γ, φ) $\mathbf{D}_{DU}$*-attacks* (Δ, ψ), if there is $\xi \in \Delta$ such that $\varphi \equiv \neg \xi$;
- (Γ, φ) $\mathbf{D}_{CU}$*-attacks* (Δ, ψ) if $\varphi \equiv \neg \bigwedge \Delta$;
- (Γ, φ) $\mathbf{D}_R$*-attacks* (Δ, ψ), if $\varphi \equiv \neg \psi$;
- (Γ, φ) $\mathbf{D}_{DR}$*-attacks* (Δ, ψ), if $\varphi \vdash \neg \psi$;

where $\varphi \equiv \psi$ means that $\varphi \vdash \psi$ and $\psi \vdash \varphi$.

The inclusive relation between the above attack relations is shown in Fig. 1.

Having defined arguments and attack relations, we can generate abstract argumentative frameworks. Such frameworks are called *premise-conclusion argumentation frameworks*.

Definition 7. Given an abstract logic $(\mathcal{L}, \vdash)$, a knowledge base Σ and an attack function $\mathbf{D}$, a *premise-conclusion argumentation framework (PAF, for short)* $\mathbf{F}_\Sigma^{\mathbf{D}}$ is a pair $(Ar_p(\Sigma), \mathbf{R_D})$, where $Ar_p(\Sigma)$ is the set of premise-conclusion arguments on Σ and $\mathbf{R_D}$ is a binary relation on $Ar_p(\Sigma)$ such that $(A, B) \in \mathbf{R_D}$ iff $\mathbf{D}(A, B) = 1$.

As abstract argumentation frameworks, PAFs can be evaluated directly using the semantics of abstract argumentation frameworks.

Example 2. Let $(\mathcal{L}, \vdash)$ be propositional logic, $\Sigma = \{p, \neg p, q\}$ and $\mathbf{D} = \mathbf{D}_{DD}$. Part of the generated PAF $\mathbf{F}_\Sigma^{\mathbf{D}_{DD}}$ is shown in Fig. 2, where nodes represent arguments, arcs represent attacks and r, s are new propositional variables.

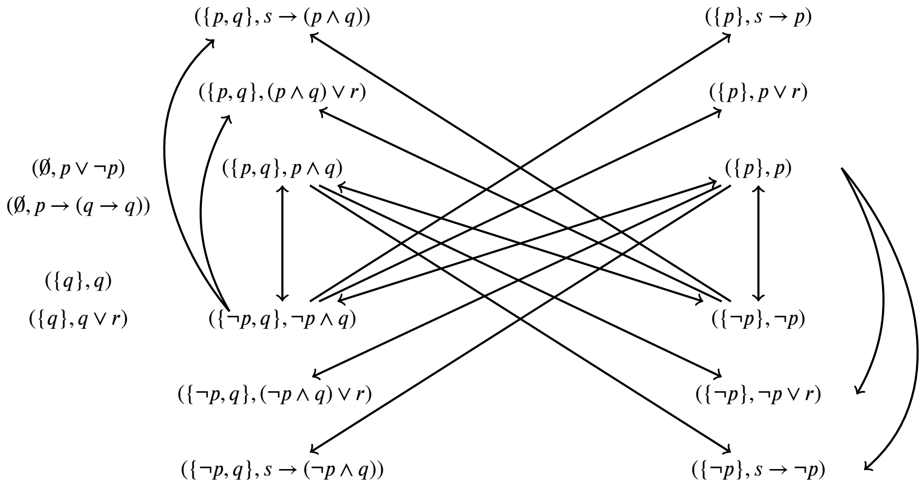


Fig. 2. Part of $\mathbf{F}_{\Sigma}^{\mathbf{D}_{DD}}$ with propositional logic as the underlying logic and $\Sigma = \{p, \neg p, q\}$.

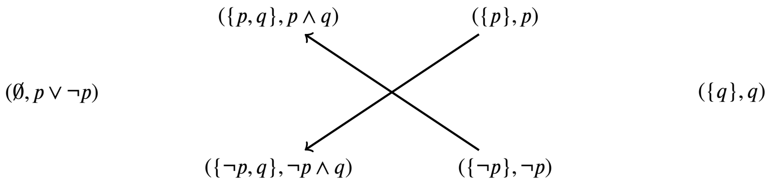


Fig. 3. Part of $\mathbf{F}_{\Sigma}^{\mathbf{D}_{DU}}$ with propositional logic as the underlying logic and $\Sigma = \{p, \neg p, q\}$.

It is worth noting that since the underlying logic is propositional logic, an infinite number of premise-conclusion arguments can be generated from one premise set. Take $\{p\}$ as an example. Since $\varphi \to (\psi \to \varphi)$ is a classical logical tautology, $(\{p\}, q \to p)$, $(\{p\}, r \to (q \to p))$, … are premise-conclusion arguments. So we have an infinite number of premise-conclusion arguments with the premise set being $\{p\}$. This explains why the above is only part of the generated PAF.

Although it is not possible to directly recognize which premise-conclusion arguments constitute complete extensions from the figure, it can be proved that the complete extensions in $\mathbf{F}_{\Sigma}^{\mathbf{D}_{DD}}$ are as follows:

$$\mathbf{S}_1 = \{A \mid \mathrm{S}(A) = \emptyset, \{q\}\},$$

$$\mathbf{S}_2 = \{A \mid \mathrm{S}(A) = \emptyset, \{q\}, \{p\} \text{ or } \{p, q\}\},$$

$$\mathbf{S}_3 = \{A \mid \mathrm{S}(A) = \emptyset, \{q\}, \{\neg p\} \text{ or } \{\neg p, q\}\}.$$

When the attack relation is changed, the premise-conclusion argument framework is changed. Let $\mathbf{D} = \mathbf{D}_{DU}$. Part of the premise-conclusion argument framework $\mathbf{F}_{\Sigma}^{\mathbf{D}_{DU}}$ is shown in Fig. 3.

3. Base argumentation

This section defines base argumentation. The basic idea is to represent an argument by its premises. Given a formal logic and a knowledge base, this means to treat some particular subsets of the knowledge base as arguments. A straightforward idea is to treat a finite consistent subset of the knowledge base as an argument. However, we find that given a consistent subset, sometimes we cannot construct a premise-conclusion argument, which goes against our intention to simulate premise-conclusion argumentation with base argumentation.

Consider a set $\{p, p \to q, q\}$. Its logical consequences are the same as $\{p, p \to q\}$. It follows that for any formula φ, if $\{p, p \to q, q\} \vdash \varphi$, then $\{p, p \to q\} \vdash \varphi$. Since $\{p, p \to q\}$ is a proper subset of $\{p, p \to q, q\}$, for any formula φ, $(\{p, p \to q, q\}, \varphi)$ does not satisfy the minimality requirement of premise-conclusion arguments.

Thus, in addition to finiteness and consistency, we also impose a minimality requirement for a base argument Γ: there is no $\Gamma' \subset \Gamma$ such that $Cn_{\vdash}(\Gamma) = Cn_{\vdash}(\Gamma')$.

Definition 8. Given a logic $(\mathcal{L}, \vdash)$ and a knowledge base Σ, a *base argument* Γ is a finite consistent subset of Σ such that there does not exist $\Gamma' \subset \Gamma$ such that $Cn_{\vdash}(\Gamma) = Cn_{\vdash}(\Gamma')$. We say that a base argument Δ is a *sub-argument* of Γ if $\Delta \subseteq \Gamma$.

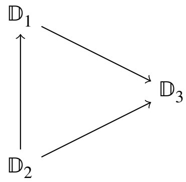


Fig. 4. Containment between attack relations on base arguments. An arrow from $\mathbb{D}_i$ to $\mathbb{D}_j$ indicates that $\mathbb{D}_i \subseteq \mathbb{D}_j$.

Note that if Γ and Δ are base arguments, $\Gamma \cup \Delta$ is not necessarily a base argument. Consider $\Gamma = \{p, p \rightarrow q\}$ and $\Delta = \{q\}$. Then $\{p, q\} \subset \Gamma \cup \Delta$ and $Cn_\vdash(\{p, q\}) = Cn_\vdash(\Gamma \cup \Delta)$. Similarly, a maximal consistent subset of Σ is not necessarily a base argument. However, if Γ and Δ are base arguments, then $\Gamma \cap \Delta$ is a base argument. This follows from the following lemma:

Lemma 2. *Given a logic $(\mathcal{L}, \vdash)$ and a knowledge base Σ, if $\Gamma \subseteq \Sigma$ is a base argument, so is any subset of Γ.*

Proof. Assume that $\Delta \subseteq \Gamma$. Since Γ is a base argument, Γ is finite, consistent, and there does not exist $\Gamma' \subset \Gamma$ such that $Cn_\vdash(\Gamma) = Cn_\vdash(\Gamma')$.

Assume that Δ is inconsistent. Then $\Delta \vdash \bot$. By the monotonicity of $\vdash$, $\Gamma \vdash \bot$, contradicting the fact that Γ is consistent. Therefore, Δ is consistent.

Assume that there exists $\Delta' \subset \Delta$ such that $Cn_\vdash(\Delta) = Cn_\vdash(\Delta')$. Since $\Delta' \subset \Delta$ and $\Delta \subseteq \Gamma$, $\Delta' \cup (\Gamma \setminus \Delta) \subset \Gamma$. Since $Cn_\vdash(\Delta) = Cn_\vdash(\Delta')$,

$$Cn_\vdash(\Delta' \cup (\Gamma \setminus \Delta)) = Cn_\vdash(\Delta \cup (\Gamma \setminus \Delta)) = Cn_\vdash(\Gamma),$$

which contradicts the fact that Γ is a base argument. It follows that there does not exist $\Delta' \subset \Delta$ such that $Cn_\vdash(\Delta) = Cn_\vdash(\Delta')$. □

Next we show that we can obtain a base argument from a premise-conclusion argument, and vice versa.

Lemma 3. *Let $(\mathcal{L}, \vdash)$ be an abstract logic and Σ a knowledge base.*

1. *For any base argument Γ, $(\Gamma, \bigwedge \Gamma)$ is a premise-conclusion argument.*
2. *For any premise-conclusion argument (Γ, φ), Γ is a base argument.*

Proof. (1) It suffices to show that there does not exist $\Gamma' \subset \Gamma$ such that $\Gamma' \vdash \bigwedge \Gamma$. We prove by contradiction and assume that $\Gamma' \vdash \bigwedge \Gamma$. It follows that $Cn_\vdash(\Gamma) \subseteq Cn_\vdash(\Gamma')$. Since $\Gamma' \subset \Gamma$, by the monotonicity of $\vdash$, $Cn_\vdash(\Gamma') \subseteq Cn_\vdash(\Gamma)$. Therefore, $Cn_\vdash(\Gamma') = Cn_\vdash(\Gamma)$, contradicting the fact that Γ is a base argument.

(2) It suffices to show that for any premise-conclusion argument (Γ, φ), there does not exist $\Gamma' \subset \Gamma$ such that $Cn_\vdash(\Gamma) = Cn_\vdash(\Gamma')$. It follows directly from the definition of premise-conclusion arguments. □

An attack relation on base arguments is represented by a function $\mathbb{D} : Ar_b(\Sigma) \times Ar_b(\Sigma) \rightarrow \{0, 1\}$, where $Ar_b(\Sigma)$ is the set of base arguments on Σ. If $\mathbb{D}(\Gamma, \Delta) = 1$, we say that Γ $\mathbb{D}$-attacks Δ. Now we define three attack relations on base arguments.

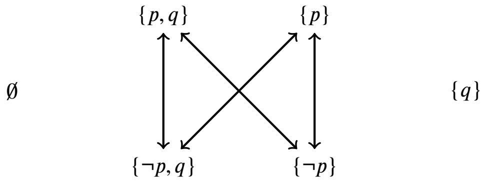
Definition 9. Given base arguments Γ, Δ,

1. Γ $\mathbb{D}_1$**-attacks** Δ, if $\Gamma \vdash \neg \bigwedge \Delta$.
2. Γ $\mathbb{D}_2$**-attacks** Δ, if there exists $\psi \in \Delta$ such that $\Gamma \vdash \neg\psi$.
3. Γ $\mathbb{D}_3$**-attacks** Δ, if there exists $\Delta' \subseteq \Delta$ such that $\Gamma \vdash \neg \bigwedge \Delta'$.

The inclusive relation between $\mathbb{D}_1, \mathbb{D}_2$ and $\mathbb{D}_3$ is shown in Fig. 4.

Definition 10. Given an abstract logic $(\mathcal{L}, \vdash)$, a knowledge base Σ and an attack function $\mathbb{D}$, a *base argumentation framework (BAF, for short)* $\mathcal{F}_\Sigma^\mathbb{D}$ is a pair $(Ar_b(\Sigma), \mathcal{R}_\mathbb{D})$, where $Ar_b(\Sigma)$ is the set of all base arguments on Σ, $\mathcal{R}_\mathbb{D}$ is a binary relation on $Ar_b(\Sigma)$ such that $\mathbb{D}(\Gamma, \Delta) = 1$ iff $(\Gamma, \Delta) \in \mathcal{R}_\mathbb{D}$.

Example 3. Let $(\mathcal{L}, \vdash)$ be propositional logic, $\Sigma = \{p, \neg p, q\}$. If $\mathbb{D} = \mathbb{D}_2$, then the generated base argumentation framework $\mathcal{F}_\Sigma^{\mathbb{D}_2}$ is as follows:



The complete extensions of $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$ are:

$\mathcal{S}_1 = \{\emptyset, \{q\}\}$,

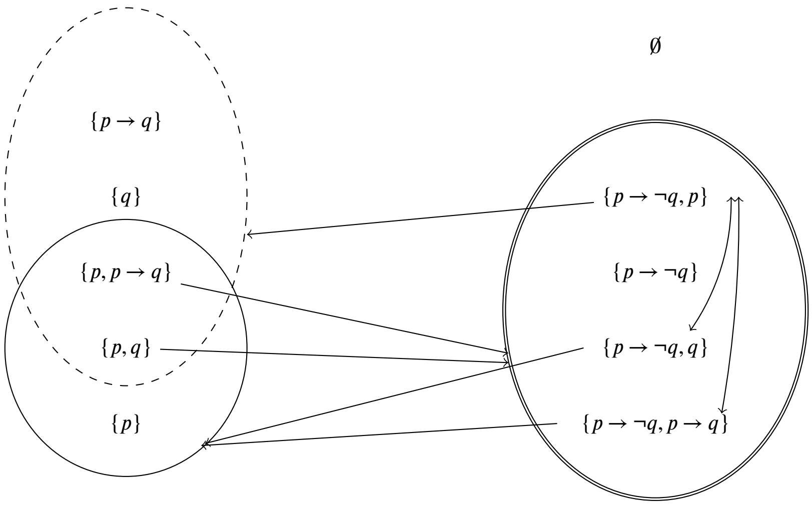
$\mathcal{S}_2 = \{\emptyset, \{p\}, \{q\}, \{p, q\}\}$,

$\mathcal{S}_2 = \{\emptyset, \{\neg p\}, \{q\}, \{\neg p, q\}\}$.

Unlike Example 2, all the arguments are shown in the figure.

Example 4. Let $(\mathcal{L}, \vdash)$ be propositional logic and $\Sigma = \{p, q, p \to q, p \to \neg q\}$. Let $\mathbb{D} = \mathbb{D}_2$. There are 16 subsets of Σ, but only 10 of them are base arguments: Since $q \to (p \to q)$ is a tautology in propositional logic, $\{q\} \vdash p \to q$ and hence $\{q, p \to q\}$, $\{p, q, p \to q\}$, $\{q, p \to q, p \to \neg q\}$ and $\{p, q, p \to q, p \to \neg q\}$ are not base arguments since they do not satisfy the minimality requirement; $\{p, q, p \to \neg q\}$ and $\{p, p \to q, p \to \neg q\}$ are not base arguments since they are inconsistent.

The base argumentation framework $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$ is as follows, where an arrow pointing to an ellipse indicates attacks on each argument in the ellipse. For example, since $\{p \to \neg q, p \to q\} \vdash \neg p$, we have $\{p \to \neg q, p \to q\}$ attacks $\{p, p \to q\}$, $\{p, q\}$ and $\{p\}$.



The complete extensions in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$ are as follows:

$\{\emptyset\}$,

$\{\emptyset, \{p\}, \{p \to \neg q, p\}, \{p \to \neg q\}\}$,

$\{\emptyset, \{p \to q\}, \{q\}, \{p, p \to q\}, \{p, q\}, \{p\}\}$,

$\{\emptyset, \{p \to q\}, \{q\}, \{p \to \neg q\}, \{p \to \neg q, q\}, \{p \to \neg q, p \to q\}\}$.

4. Extensional equivalence

Comparing base argumentation with premise-conclusion argumentation, it is evident that base argumentation has the advantage of simplicity in formal representation of knowledge. As a more concise tool, if base argumentation has the same function as premise-conclusion argumentation to some extent, it will be more scientifically meaningful. As formal argumentation aims to differentiate acceptable arguments from unacceptable ones, we evaluate whether base argumentation and premise-conclusion argumentation have the same function from this perspective.

Assume as given an abstract logic $(\mathcal{L}, \vdash)$ and a knowledge base Σ. Fix an attack relation $\mathbb{D}$ on base arguments and an attack function $\mathbf{D}$ on premise-conclusion arguments. Then we have a BAF $\mathcal{F}_{\Sigma}^{\mathbb{D}} = (Ar_b(\Sigma), \mathcal{R}_{\mathbb{D}})$ and a PAF $\mathbf{F}_{\Sigma}^{\mathbf{D}} = (Ar_p(\Sigma), \mathbf{R}_{\mathbf{D}})$. Consider an extensional semantics X. The following statement illustrates our intuition about when two argumentation systems have the same ability in discerning acceptable arguments: If a BAF $\mathcal{F}_{\Sigma}^{\mathbb{D}}$ and a PAF $\mathbf{F}_{\Sigma}^{\mathbf{D}}$ accept the 'same' arguments, then for each X-extension $\mathcal{S}$ in $\mathcal{F}$, there exists an X-extension $\mathbf{S}$ in $\mathbf{F}$ such that $\mathcal{S}$ and $\mathbf{S}$ are the 'same', and vice versa.

What remains to be done is to define when a set of base arguments and a set of premise-conclusion arguments are the 'same'.

Note that each X-extension in $\mathbb{F}$ is of the form

$$\mathcal{S} = \{\Gamma_1, \ldots, \Gamma_m\},$$

and each X-extension in $\mathbf{F}$ is of the form

$$\mathbf{S} = \{(\Gamma_1', \varphi_1), \ldots, (\Gamma_n', \varphi_n)\}.$$

If $\mathcal{S}$ and $\mathbf{S}$ are the 'same', then being the same can not be defined in terms of set-theoretical identity. Recall that our slogan for base argumentation (the logical omniscience hypothesis) is that each argument can be represented by its premises. Let us define an operation $(.)^{\bullet}$ that collects all the premise sets in $\mathbf{S}$, and an operation $(.)_{\bullet}$ that builds all possible premise-conclusion arguments from $\mathcal{S}$. If $\mathbf{S}^{\bullet} = \mathcal{S}$ and $\mathcal{S}_{\bullet} = \mathbf{S}$, by the assumption that each argument can be represented by its premises, we can say that $\mathcal{S}$ and $\mathbf{S}$ are the same.

Now we have an informal definition for a BAF and a PAF to have the same ability in discerning acceptable arguments. In this case, we say that they are *X-extensionally equivalent*. The remaining task in this section is to formalize the above ideas.

Transformation functions First we define operations $(.)_{\bullet}$ and $(.)^{\bullet}$. They are called *transformation functions*.

Definition 11. [Transformation functions] Let $(\mathcal{L}, \vdash)$ be an abstract logic and Σ a knowledge base. Let $\mathcal{F}_{\Sigma}^{\mathbb{D}} = (Ar_b(\Sigma), \mathcal{R}_{\mathbb{D}})$ be a BAF, and $\mathbf{F}_{\Sigma}^{\mathbf{D}} = (Ar_p(\Sigma), \mathbf{R}_{\mathbf{D}})$ a PAF. Let $\mathcal{S} \subseteq Ar_b(\Sigma)$, $\mathbf{S} \subseteq Ar_p(\Sigma)$.

1. Define a function $(.)_{\bullet} : \mathcal{P}(Ar_b(\Sigma)) \to \mathcal{P}(Ar_p(\Sigma))$ as follows:

$$\mathcal{S}_{\bullet} := \{(\Gamma, \varphi) \mid \Gamma \in \mathcal{S} \text{ and } (\Gamma, \varphi) \in Ar_p(\Sigma)\}.$$

2. Define a function $(.)^{\bullet} : \mathcal{P}(Ar_p(\Sigma)) \to \mathcal{P}(Ar_b(\Sigma))$ as follows:

$$\mathbf{S}^{\bullet} := \{\Gamma \mid \exists \varphi, (\Gamma, \varphi) \in \mathbf{S}\}.$$

For a set of base arguments $\mathcal{S}$, $\mathcal{S}_{\bullet}$ is the set of premise-conclusion arguments that can be generated by these base arguments; for a set of premise-conclusion arguments $\mathbf{S}$, $\mathbf{S}^{\bullet}$ is the set of premises of premise-conclusion arguments in $\mathbf{S}$. By Lemma 3, each element in $\mathbf{S}^{\bullet}$ is a base argument.

Functions $(.)_{\bullet}$ and $(.)^{\bullet}$ reflect the logical omniscience hypothesis from two perspectives: the former reflects that if one knows a set of formulas, then he/she knows all its logical consequences, and the latter reflects that for a set of premise-conclusion arguments, an agent only needs to know their premises.

X-extensional equivalence Now we define the notion of *X-extensional equivalence*.

Given a BAF $\mathcal{F}_{\Sigma}^{\mathbb{D}}$ and a PAF $\mathbf{F}_{\Sigma}^{\mathbf{D}}$, for extension semantics X, we use $\mathcal{E}_X(\mathcal{F}_{\Sigma}^{\mathbb{D}})$ for the set of X-extensions in $\mathcal{F}_{\Sigma}^{\mathbb{D}}$ and $\mathbf{E}_X(\mathbf{F}_{\Sigma}^{\mathbf{D}})$ for the set of X-extension in $\mathbf{F}_{\Sigma}^{\mathbf{D}}$. We adopt the following abbreviations: 'co' for 'complete', 'st' for 'stable', 'gr' for 'grounded', 'pr' for 'preferred', 'ss' for 'semi-stable', 'id' for 'ideal' and 'ea' for 'eager'. For example, $\mathcal{E}_{co}(\mathcal{F}_{\Sigma}^{\mathbb{D}})$ is the set of complete extensions in $\mathcal{F}_{\Sigma}^{\mathbb{D}}$.

Definition 12. Given a logic $(\mathcal{L}, \vdash)$ and a knowledge base Σ. Let $\mathcal{F}_{\Sigma}^{\mathbb{D}} = (Ar_b(\Sigma), \mathcal{R}_{\mathbb{D}})$ be a BAF and $\mathbf{F}_{\Sigma}^{\mathbf{D}} = (Ar_p(\Sigma), \mathbf{R}_{\mathbf{D}})$ a PAF. $\mathcal{F}_{\Sigma}^{\mathbb{D}}$ and $\mathbf{F}_{\Sigma}^{\mathbf{D}}$ are said to be *X-extensionally equivalent*, if

1. When restricted to $\mathcal{E}_X(\mathcal{F}_{\Sigma}^{\mathbb{D}})$, $(.)_{\bullet}$ is a bijection from $\mathcal{E}_X(\mathcal{F}_{\Sigma}^{\mathbb{D}})$ to $\mathbf{E}_X(\mathbf{F}_{\Sigma}^{\mathbf{D}})$.
2. When restricted to $\mathbf{E}_X(\mathbf{F}_{\Sigma}^{\mathbf{D}})$, $(.)^{\bullet}$ is a bijection from $\mathbf{E}_X(\mathbf{F}_{\Sigma}^{\mathbf{D}})$ to $\mathcal{E}_X(\mathcal{F}_{\Sigma}^{\mathbb{D}})$.

If a BAF and a PAF are X-extensionally equivalent, they have the same number of X-extensions. In addition, their extensions are logically related in terms of content due to the logical connotation of $(.)^{\bullet}$ and $(.)_{\bullet}$.

Example 5. In Examples 2 and 3, $(\mathcal{L}, \vdash)$ is propositional logic and Σ is $\{p, \neg p, q\}$. Let $\mathbf{D} = \mathbf{D}_{DD}$ and $\mathbb{D} = \mathbb{D}_2$. By the analysis in the above examples, the complete extensions in $\mathbf{F}_{\Sigma}^{\mathbf{D}_{DD}}$ are

$$\mathbf{S}_1 = \{A \mid \mathrm{S}(A) = \emptyset, \{q\}\},$$

$$\mathbf{S}_2 = \{A \mid \mathrm{S}(A) = \emptyset, \{q\}, \{p\} \text{ or } \{p, q\}\},$$

$$\mathbf{S}_3 = \{A \mid \mathrm{S}(A) = \emptyset, \{q\}, \{\neg p\} \text{ or } \{\neg p, q\}\}.$$

The complete extensions in $\mathcal{F}_\Sigma^{\mathbb{D}_2}$ are $\mathcal{S}_1 = \{\emptyset, \{q\}\}$, $\mathcal{S}_2 = \{\emptyset, \{p\}, \{q\}, \{p,q\}\}$, $\mathcal{S}_2 = \{\emptyset, \{\neg p\}, \{q\}, \{\neg p, q\}\}$.

It follows that

$$\mathbf{S}_1^\bullet = \mathcal{S}_1 \quad (\mathcal{S}_1)_\bullet = \mathbf{S}_1$$
$$\mathbf{S}_2^\bullet = \mathcal{S}_2 \quad (\mathcal{S}_2)_\bullet = \mathbf{S}_2$$
$$\mathbf{S}_3^\bullet = \mathcal{S}_3 \quad (\mathcal{S}_3)_\bullet = \mathbf{S}_3.$$

Hence, $(.)^\bullet : \mathbf{E}_{co}(\mathbf{F}_\Sigma^{\mathbf{D}_{DD}}) \to \mathcal{E}_{co}(\mathcal{F}_\Sigma^{\mathbb{D}_2})$ and $(.)_\bullet : \mathcal{E}_{co}(\mathcal{F}_\Sigma^{\mathbb{D}_2}) \to \mathbf{E}_{co}(\mathbf{F}_\Sigma^{\mathbf{D}_{DD}})$ are bijections. It follows that $\mathcal{F}_\Sigma^{\mathbb{D}_2}$ and $\mathbf{F}_\Sigma^{\mathbf{D}_{DD}}$ are complete-extensionally equivalent.

5. Bisimulation

With the notion of extensional equivalence defined, an important question is under what conditions a BAF and a PAF are extensionally equivalent. This section defines the notion of *bisimulation*, and the next section shows that bisimulation implies extensional equivalence.

The notion of bisimulation between a BAF and a PAF is similar to the notion of *bisimulation in transition systems* which describes the behavioral equivalences between transition systems (see e.g., Section 2.3.4 in [21]). Given a labeled transition system $(Q, A, \to)$, where Q is a nonempty set of states, A is a countable set of labels and $\to \subseteq Q \times A \times Q$ is the transition relation, a *bisimulation* is a binary relation $R \subseteq S \times S$ such that if $(q_1, q_2) \in R$, then for all $\mu \in A$,

(1) for all q_1' such that $q_1 \xrightarrow{\mu} q_1'$, there exists a state q_2' such that $q_2 \xrightarrow{\mu} q_2'$ and $(q_1', q_2') \in R$;
(2) for all q_2' such that $q_2 \xrightarrow{\mu} q_2'$, there exists a state q_1' such that $q_1 \xrightarrow{\mu} q_1'$ and $(q_1', q_2') \in R$.

Now we give the definition of bisimulation between a BAF and a PAF.

Definition 13. [Bisimulation] Let $(\mathcal{L}, \vdash)$ be an abstract logic and Σ a knowledge base. Let $\mathcal{F}_\Sigma^{\mathbb{D}} = (Ar_b(\Sigma), \mathcal{R}_\mathbb{D})$ be a BAF and $\mathbf{F}_\Sigma^{\mathbf{D}} = (Ar_p(\Sigma), \mathbf{R_D})$ a PAF. $\mathcal{F}_\Sigma^{\mathbb{D}}$ and $\mathbf{F}_\Sigma^{\mathbf{D}}$ are said to be *bisimilar*, if the following conditions are satisfied:

(1) For $\Gamma, \Delta \in Ar_b(\Sigma)$, if Γ $\mathbb{D}$-attacks Δ, then there exists $(\Gamma', \varphi), (\Delta, \psi) \in Ar_p(\Sigma)$ such that $\Gamma' \subseteq \Gamma$ and (Γ', φ) **D**-attacks (Δ, ψ).
(2) For $(\Gamma, \varphi), (\Delta, \psi) \in Ar_p(\Sigma)$, if (Γ, φ) **D**-attacks (Δ, ψ), then Γ $\mathbb{D}$-attacks Δ.
(3) $\mathbb{D}$ is *monotonic*: if $\mathbb{D}(\Gamma, \Delta) = 1$ and $\Delta \subseteq \Omega \in Ar_b(\Sigma)$, then $\mathbb{D}(\Gamma, \Omega) = 1$.
(4) **D** is *p-monotonic*: if $\mathbf{D}(A, B) = 1$ and $\mathrm{S}(B) \subseteq \mathrm{S}(C)$, then $\mathbf{D}(A, C) = 1$.

Condition (2) is well-defined because the premise set of a premise-conclusion argument is a base argument (Lemma 3).

Bisimulation imposes specific requirements on $\mathcal{R}_\mathbb{D}$ and $\mathbf{R_D}$. Condition (1) says that $\mathbf{R_D}$ is able to simulate $\mathcal{R}_\mathbb{D}$ and condition (2) says that $\mathcal{R}_\mathbb{D}$ is able to simulate $\mathbf{R_D}$. For an attack function $\mathbb{D}$ on base arguments, being monotonic means that if Γ $\mathbb{D}$-attacks Δ, then Γ $\mathbb{D}$-attacks any base argument that is a superset of Δ. For an attack function **D** on premise-conclusion arguments, being p-monotonic (short for 'monotonic on premises') means that if (Γ, φ) $\mathbb{D}$-attacks (Δ, ψ), then (Γ, φ) **D**-attacks any premise-conclusion argument whose premise set is a superset of Δ. It implies that whether a premise-conclusion argument is **D**-attacked is completely determined by its premise set. Conditions (3) and (4) ensure the property of closure under sub-arguments (Lemma 4), which is frequently used in subsequent proofs. For example, it is used to show that for any $\mathbf{S} \subseteq Ar_p(\Sigma)$,

if $\mathbf{S}$ is a complete extension in $\mathbf{F}_\Sigma^{\mathbf{D}}$, then $(\mathbf{S}^\bullet)_\bullet = \mathbf{S}$ (Item (4) in Lemma 6),

which implies that $(.)_\bullet$ is a surjection if restricted to complete extensions (Proposition 4).

Example 6. The PAF $\mathbf{F}_\Sigma^{\mathbf{D}_{DD}}$ in Example 2 and the BAF $\mathcal{F}_\Sigma^{\mathbb{D}_2}$ in Example 3 are bisimilar. See Proposition 2 for a proof.

The following proposition shows that some frameworks with attack relations in Definitions 6 and 9 are bisimilar.

Proposition 2. *Let $(\mathcal{L}, \vdash)$ be a logic and Σ a knowledge base.*

(1) $\mathcal{F}_\Sigma^{\mathbb{D}_1}$ and $\mathbf{F}_\Sigma^{\mathbf{D}_D}$ are bisimilar.
(2) $\mathcal{F}_\Sigma^{\mathbb{D}_2}$ and $\mathbf{F}_\Sigma^{\mathbf{D}_{DD}}$ are bisimilar.
(3) $\mathcal{F}_\Sigma^{\mathbb{D}_2}$ and $\mathbf{F}_\Sigma^{\mathbf{D}_{DU}}$ are bisimilar.
(4) $\mathcal{F}_\Sigma^{\mathbb{D}_3}$ and $\mathbf{F}_\Sigma^{\mathbf{D}_U}$ are bisimilar.

Proof. (1) Assumes that Γ $\mathbb{D}_1$-attacks Δ. Then $\Gamma \vdash \neg \bigwedge \Delta$.

Let Γ' be a minimal consistent subset of Γ such that $\Gamma' \vdash \neg \bigwedge \Delta$. Therefore, premise-conclusion argument $(\Gamma', \neg \bigwedge \Delta)$ $\mathbf{D}_D$-attacks $(\Delta, \bigwedge \Delta)$.

Assume that (Γ, φ) $\mathbf{D}_D$-attacks (Δ, ψ). Then $\varphi \vdash \neg \bigwedge \Delta$. Since (Γ, φ) is a premise-conclusion argument, $\Gamma \vdash \varphi$. By the transitivity of $\vdash$, $\Gamma \vdash \neg \bigwedge \Delta$. It follows that Γ $\mathbb{D}_1$-attacks Δ.

It is straightforward that $\mathbb{D}_1$ is monotonic and that $\mathbf{D}_D$ is p-monotonic.

(2) Since Γ $\mathbb{D}_2$- attacks Δ, there exists $\psi \in \Delta$ such that $\Gamma \vdash \neg\psi$. Let Γ' be a minimal consistent subset of Γ such that $\Gamma' \vdash \neg\psi$. It follows that premise-conclusion argument $(\Gamma', \neg\psi)$ $\mathbf{D}_{DD}$-attacks $(\Delta, \bigwedge \Delta)$.

Assume that (Γ, φ) $\mathbf{D}_{DD}$-attacks (Δ, ψ). Then there exists $\varphi \in \Delta$ such that $\varphi \vdash \neg\varphi$. By the transitivity of $\vdash$, $\Gamma \vdash \neg\varphi$. Therefore, Γ $\mathbb{D}_2$-attacks Δ.

It is straightforward that $\mathbb{D}_2$ is monotonic and that $\mathbf{D}_{DD}$ is p-monotonic.

(3) Assume that Γ $\mathbb{D}_2$-attacks Δ. Then there exists $\psi \in \Delta$ such that $\Gamma \vdash \neg\psi$. Let Γ' be a minimal consistent subset of Γ such that $\Gamma' \vdash \neg\psi$. It follows that premise-conclusion argument $(\Gamma', \neg\psi)$ $\mathbf{D}_{DU}$-attacks $(\Delta, \bigwedge \Delta)$.

Assume that (Γ, φ) $\mathbf{D}_{DU}$-attacks (Δ, ψ). Then there exists $\varphi \in \Delta$ such that $\varphi \equiv \neg\varphi$. By the transitivity of $\vdash$, $\Gamma \vdash \neg\varphi$. Therefore, Γ $\mathbb{D}_2$-attacks Δ.

It is straightforward that $\mathbb{D}_2$ is monotonic and that $\mathbf{D}_{DU}$ is p-monotonic.

(4) Assume that Γ $\mathbb{D}_3$-attacks Δ. Then there exists $\Delta' \subseteq \Delta$ such that $\Gamma \vdash \neg \bigwedge \Delta'$. Let Γ' be a minimal consistent subset of Γ such that $\Gamma' \vdash \neg \bigwedge \Delta'$. It follows that premise-conclusion argument $(\Gamma', \neg \bigwedge \Delta')$ $\mathbf{D}_U$-attacks $(\Delta, \bigwedge \Delta)$.

Assume that (Γ, φ) $\mathbf{D}_U$-attacks (Δ, ψ). Then there exists $\Delta' \subseteq \Delta$ such that $\varphi \equiv \neg \bigwedge \Delta'$. By the monotonicity of $\vdash$, $\Gamma \vdash \neg \bigwedge \Delta'$. Therefore, Γ $\mathbb{D}_3$-attacks Δ.

It is straightforward that $\mathbb{D}_3$ is monotonic and that $\mathbf{D}_U$ is p-monotonic. □

Non-monotonicity prevents bisimilarity for all other types of attack. For $\mathcal{F}_\Sigma^{\mathbb{D}_1}$ and $\mathbf{F}_\Sigma^{\mathbf{D}_{CU}}$, since $\mathbf{D}_{CU}$ is not p-monotonic, they are not bisimilar, though they satisfy conditions (1)-(3). For the same reason, $\mathcal{F}_\Sigma^{\mathbb{D}_1}$ and $\mathbf{F}_\Sigma^{\mathbf{D}_R}$ are not bisimilar, and $\mathcal{F}_\Sigma^{\mathbb{D}_1}$ and $\mathbf{F}_\Sigma^{\mathbf{D}_{DR}}$ are not bisimilar.

Items (2) and (3) in Proposition 2 show that it is possible that a BAF simulates more than one PAF.

6. Bisimulation implies extensional equivalence

This section shows that bisimulation between a BAF $\mathcal{F}_\Sigma^{\mathbb{D}}$ and a PAF $\mathbf{F}_\Sigma^{\mathbf{D}}$ implies X-extensional equivalence, for $X \in \{co, st, gr, pr, ss, id, ea\}$. This is achieved by showing that $(.)_\bullet$ and $(.)^\bullet$ satisfies the following conditions:

(1) for each X-extension $\mathcal{S}$ in $\mathcal{F}_\Sigma^{\mathbb{D}}$, $\mathcal{S}_\bullet$ is an X-extension in $\mathbf{F}_\Sigma^{\mathbf{D}}$;
(2) for each X-extension $\mathbf{S}$ in $\mathbf{F}_\Sigma^{\mathbf{D}}$, $\mathbf{S}^\bullet$ is an X-extension in $\mathcal{F}_\Sigma^{\mathbb{D}}$;
(3) $(.)_\bullet$ and $(.)^\bullet$ are bijections when restricted X-extensions.

For the results in this section, it is assumed as given an abstract logic $(\mathcal{L}, \vdash)$ and a knowledge base Σ. Moreover, when the attack relation is clear from the context, we remove the prefix $\mathbf{D}$ or $\mathbb{D}$ for simplicity. For example, we write "premise-conclusion argument (Γ, φ) $\mathbf{D}$-attacks (Δ, ψ)" as "premise-conclusion argument (Γ, φ) attacks (Δ, ψ)" and write "base argument Γ $\mathbb{D}$-attacks Δ" as "base argument Γ attacks Δ".

6.1. Key lemmas

This subsection proves lemmas that will be frequently used. The first one shows that complete extensions in bisimilar frames are closed under sub-arguments.

Lemma 4. *[Closure under sub-arguments] Let $\mathcal{F}_\Sigma^{\mathbb{D}} = (Ar_b(\Sigma), \mathcal{R}_{\mathbb{D}})$ be a BAF and $\mathbf{F}_\Sigma^{\mathbf{D}} = (Ar_p(\Sigma), \mathbf{R}_{\mathbf{D}})$ a PAF such that $\mathcal{F}_\Sigma^{\mathbb{D}}$ and $\mathbf{F}_\Sigma^{\mathbf{D}}$ are bisimilar.*

(1) For a complete extension $\mathbf{S}$ in $\mathbf{F}_\Sigma^{\mathbf{D}}$, if $(\Gamma, \varphi) \in \mathbf{S}$, $(\Delta, \psi) \in Ar_p(\Sigma)$ and $\Delta \subseteq \Gamma$, then $(\Delta, \psi) \in \mathbf{S}$.
(2) For a complete extension $\mathcal{S}$ in $\mathcal{F}_\Sigma^{\mathbb{D}}$, if $\Gamma \in \mathcal{S}$ and $\Delta \subseteq \Gamma$, then $\Delta \in \mathcal{S}$.

Proof. (1) Since $\mathbf{S}$ is a complete extension, to show that $(\Delta, \psi) \in \mathbf{S}$, it suffices to show that $\mathbf{S}$ defends (Δ, ψ). Let (Ξ, φ) be a premise-conclusion argument that attacks (Δ, ψ). Since $\mathbf{D}$ is p-monotonic and $\Delta \subseteq \Gamma$, (Ξ, φ) attacks (Γ, φ).

Since $(\Gamma, \varphi) \in \mathbf{S}$ and $\mathbf{S}$ is complete, $\mathbf{S}$ attacks (Ξ, φ). Therefore, $\mathbf{S}$ defends (Δ, ψ).

(2) By Lemma 2, Δ is a base argument. It suffices to show that $\mathcal{S}$ defends Δ. Assume that a base argument Ω attacks Δ. Since $\mathbb{D}$ is monotonic and $\Delta \subseteq \Gamma$, Ω attacks Γ. Since $\Gamma \in \mathcal{S}$ and $\mathcal{S}$ is a complete extension, $\mathcal{S}$ attacks Ω. □

The following lemma is a direct consequence of p-monotonicity and condition (1) in the definition of bisimulation (Definition 13).

Lemma 5. *Let $\mathcal{F}_\Sigma^{\mathbb{D}} = (Ar_b(\Sigma), \mathcal{R}_{\mathbb{D}})$ be a BAF and $\mathbf{F}_\Sigma^{\mathbf{D}} = (Ar_p(\Sigma), \mathbf{R}_{\mathbf{D}})$ a PAF such that $\mathcal{F}_\Sigma^{\mathbb{D}}$ and $\mathbf{F}_\Sigma^{\mathbf{D}}$ are bisimilar.*

Let $\Gamma, \Delta \in Ar_b(\Sigma)$. If Γ $\mathbb{D}$-attacks Δ, then there exists a premise-conclusion argument (Γ', φ) with $\Gamma' \subseteq \Gamma$ which **D***-attacks any premise-conclusion argument whose premise set is Δ.*

Proof. Since $\mathcal{F}_\Sigma^{\mathbb{D}}$ and $\mathbf{F}_\Sigma^{\mathbf{D}}$ are bisimilar and Γ $\mathbb{D}$-attacks Δ, by definition, there exist premise-conclusion arguments (Γ', φ) and (Δ, ψ), where $\Gamma' \subseteq \Gamma$, such that (Γ', φ) **D**-attacks (Δ, ψ). Since **D** is p-monotonic, (Γ', φ) **D**-attacks any premise-conclusion argument whose premise set is Δ. □

The following lemma is about the interaction of functions $(.)_\bullet$, $(.)^\bullet$ and other operations.

Lemma 6. *Let $\mathcal{F}_\Sigma^{\mathbb{D}} = (Ar_b(\Sigma), \mathcal{R}_{\mathbb{D}})$ be a BAF and $\mathbf{F}_\Sigma^{\mathbf{D}} = (Ar_p(\Sigma), \mathbf{R_D})$ a PAF such that $\mathcal{F}_\Sigma^{\mathbb{D}}$ and $\mathbf{F}_\Sigma^{\mathbf{D}}$ are bisimilar. Let $\mathcal{S}, \mathcal{S}' \subseteq Ar_b(\Sigma)$ and $\mathbf{S}, \mathbf{S}' \subseteq Ar_p(\Sigma)$.*

(1) If $\mathcal{S} \subseteq \mathcal{S}'$, then $\mathcal{S}_\bullet \subseteq \mathcal{S}'_\bullet$.
(2) If $\mathbf{S} \subseteq \mathbf{S}'$, then $\mathbf{S}^\bullet \subseteq (\mathbf{S}')^\bullet$.
(3) $(\mathcal{S}_\bullet)^\bullet = \mathcal{S}$.
(4) If $\mathbf{S}$ is a complete extension in $\mathbf{F}_\Sigma^{\mathbf{D}}$, then $(\mathbf{S}^\bullet)_\bullet = \mathbf{S}$.
(5) $(\mathcal{S} \cup \mathcal{S}')_\bullet = \mathcal{S}_\bullet \cup (\mathcal{S}')_\bullet$.
(6) $(\mathbf{S} \cup \mathbf{S}')^\bullet = \mathbf{S}^\bullet \cup (\mathbf{S}')^\bullet$.
(7) If $\mathcal{S}$ is a complete extension in $\mathcal{F}_\Sigma^{\mathbb{D}}$, then $(\mathcal{S}_\bullet)^+ = (\mathcal{S}^+)_\bullet$.
(8) If $\mathbf{S}$ is a complete extension in $\mathbf{F}_\Sigma^{\mathbf{D}}$, then $(\mathbf{S}^+)^\bullet = (\mathbf{S}^\bullet)^+$.
(9) For a set $\mathcal{E}_c$ of complete extensions in $\mathcal{F}_\Sigma^{\mathbb{D}}$, $(\bigcap_{\mathcal{S}\in\mathcal{E}_c} \mathcal{S})_\bullet = \bigcap_{\mathcal{S}\in\mathcal{E}_c} \mathcal{S}_\bullet$.
(10) For a set $\mathbf{E}_c$ of complete extensions in $\mathbf{F}_\Sigma^{\mathbf{D}}$, $(\bigcap_{\mathbf{S}\in\mathbf{E}_c} \mathbf{S})^\bullet = \bigcap_{\mathbf{S}\in\mathbf{E}_c} \mathbf{S}^\bullet$.

Proof. Items (1) and (2) follow directly from the definitions.
(3) By definition,

$$(\mathcal{S}_\bullet)^\bullet = \{\Gamma \mid \exists\varphi, (\Gamma, \varphi) \in Ar_p(\Sigma) \text{ and } \Gamma \in \mathcal{S}\}.$$

It follows from the definition that $(\mathcal{S}_\bullet)^\bullet \subseteq \mathcal{S}$.
For the other direction, assume that $\Gamma \in \mathcal{S}$. By Lemma 3, $(\Gamma, \bigwedge \Gamma)$ is a premise-conclusion argument. Therefore, $\Gamma \in (\mathcal{S}_\bullet)^\bullet$.
(4) By definition,

$$(\mathbf{S}^\bullet)_\bullet = \{(\Gamma, \varphi) \mid (\Gamma, \varphi) \in Ar_p(\Sigma) \text{ and } \exists\psi(\Gamma, \psi) \in \mathbf{S}\}.$$

It follows from the definition that $\mathbf{S} \subseteq (\mathbf{S}^\bullet)_\bullet$.
Assume that $(\Gamma, \varphi) \in (\mathbf{S}^\bullet)_\bullet$. Then (Γ, φ) is a premise-conclusion argument and there exists ψ such that $(\Gamma, \psi) \in \mathbf{S}$.
By Lemma 4, $(\Gamma, \varphi) \in \mathbf{S}$.
Items (5) and (6) follow directly from the definition.
(7) By definition,

$$(\mathcal{S}_\bullet)^+ = \{(\Gamma, \varphi) \mid \mathcal{S}_\bullet \text{ attacks } (\Gamma, \varphi)\},$$
$$(\mathcal{S}^+)_\bullet = \{(\Gamma, \varphi) \mid (\Gamma, \varphi) \in Ar_p(\Sigma) \text{ and } \Gamma \in \mathcal{S}^+\}.$$

Assume that $(\Gamma, \varphi) \in (\mathcal{S}_\bullet)^+$. Then $\mathcal{S}_\bullet$ attacks (Γ, φ). Then there exists $(\Delta, \psi) \in \mathcal{S}_\bullet$ that attacks (Γ, φ). Since $\mathcal{F}_\Sigma^{\mathbb{D}}$ and $\mathbf{F}_\Sigma^{\mathbf{D}}$ are bisimilar, Δ attacks Γ. Since $(\Delta, \psi) \in \mathcal{S}_\bullet$, $\Delta \in \mathcal{S}$. It follows that $\Gamma \in \mathcal{S}^+$. Therefore, $(\Gamma, \varphi) \in (\mathcal{S}^+)_\bullet$. Hence, $(\mathcal{S}_\bullet)^+ \subseteq (\mathcal{S}^+)_\bullet$.
Assume that $(\Gamma, \varphi) \in (\mathcal{S}^+)_\bullet$. Then $\Gamma \in \mathcal{S}^+$. It follows that there exists $\Delta \in \mathcal{S}$ that attacks Γ. Since $\mathcal{F}_\Sigma^{\mathbb{D}}$ and $\mathbf{F}_\Sigma^{\mathbf{D}}$ are bisimilar, by Lemma 5, there exists premise-conclusion argument (Δ', φ) that attacks (Γ, φ), where $\Delta' \subseteq \Delta$.
Since $\Delta \in \mathcal{S}$, by Lemma 4, $\Delta' \in \mathcal{S}$. Therefore, $(\Delta', \varphi) \in \mathcal{S}_\bullet$. It follows that $(\Gamma, \varphi) \in (\mathcal{S}_\bullet)^+$. Hence, $(\mathcal{S}^+)_\bullet \subseteq (\mathcal{S}_\bullet)^+$.
(8) By definition,

$$(\mathbf{S}^+)^\bullet = \{\Gamma \mid \text{ there exists } \varphi \text{ such that } (\Gamma, \varphi) \in \mathbf{S}^+\},$$
$$(\mathbf{S}^\bullet)^+ = \{\Gamma \mid \mathbf{S}^\bullet \text{ attacks } \Gamma\}.$$

Assume that $\Gamma \in (\mathbf{S}^+)^\bullet$. Then there exists φ such that $(\Gamma, \varphi) \in \mathbf{S}^+$. It follows that there exists $(\Delta, \psi) \in \mathbf{S}$ that attacks (Γ, φ). Since $\mathcal{F}_\Sigma^{\mathbb{D}}$ and $\mathbf{F}_\Sigma^{\mathbf{D}}$ are bisimilar, Δ attacks Γ. Since $(\Delta, \psi) \in \mathbf{S}$, $\Delta \in \mathbf{S}^\bullet$. Therefore, $\mathbf{S}^\bullet$ attacks Γ.
Assume that $\Gamma \in (\mathbf{S}^\bullet)^+$. Then $\mathbf{S}^\bullet$ attacks Γ. Then there exists $\Delta \in \mathbf{S}^\bullet$ that attacks Γ.
By Lemma 3, $(\Gamma, \bigwedge \Gamma)$ is a premise-conclusion argument. Since $\mathcal{F}_\Sigma^{\mathbb{D}}$ and $\mathbf{F}_\Sigma^{\mathbf{D}}$ are bisimilar, by Lemma 5, there exists premise-conclusion argument (Δ', φ) that attacks $(\Gamma, \bigwedge \Gamma)$, where $\Delta' \subseteq \Delta$.
Since $\Delta \in \mathbf{S}^\bullet$, there exists ψ such that $(\Delta, \psi) \in \mathbf{S}$.
By Lemma 4, $(\Delta', \varphi) \in \mathbf{S}$. Since (Δ', φ) attacks $(\Gamma, \bigwedge \Gamma)$, $(\Gamma, \bigwedge \Gamma) \in \mathbf{S}^+$. Therefore, $\Gamma \in (\mathbf{S}^+)^\bullet$.

(9) By definition,

$$(\bigcap_{\mathcal{S}\in\mathcal{E}_c}\mathcal{S})_\bullet = \{(\Gamma,\varphi) \mid (\Gamma,\varphi)\in Ar_p(\Sigma) \text{ and } \Gamma\in\bigcap_{\mathcal{S}\in\mathcal{E}_c}\mathcal{S}\},$$

$$\bigcap_{\mathcal{S}\in\mathcal{E}_c}\mathcal{S}_\bullet = \{(\Gamma,\varphi) \mid (\Gamma,\varphi)\in Ar_p(\Sigma) \text{ and } \forall \mathcal{S}\in\mathcal{E}_c, \Gamma\in\mathcal{S}\}.$$

It follows directly from the definition that $(\bigcap_{\mathcal{S}\in\mathcal{E}_c}\mathcal{S})_\bullet = \bigcap_{\mathcal{S}\in\mathcal{E}_c}\mathcal{S}_\bullet$.

(10) By definition,

$$(\bigcap_{\mathbf{S}\in\mathbf{E}_c}\mathbf{S})^\bullet = \{\Gamma \mid \text{there exists } \varphi \text{ such that } (\Gamma,\varphi)\in\bigcap_{\mathbf{S}\in\mathbf{E}_c}\mathbf{S}\},$$

$$\bigcap_{\mathbf{S}\in\mathbf{E}_c}\mathbf{S}^\bullet = \{\Gamma \mid \text{for any } \mathbf{S}\in\mathbf{E}_c, \text{ there exists } \varphi \text{ such that } (\Gamma,\varphi)\in\mathbf{S}\}.$$

It follows from the definition that $(\bigcap_{\mathbf{S}\in\mathbf{E}_c}\mathbf{S})^\bullet \subseteq \bigcap_{\mathbf{S}\in\mathbf{E}_c}\mathbf{S}^\bullet$.

Assume that $\Gamma\in\bigcap_{\mathbf{S}\in\mathbf{E}_c}\mathbf{S}^\bullet$. Then for any $\mathbf{S}\in\mathbf{E}_c$, there exists $\varphi_{\mathbf{S}}$ such that $(\Gamma,\varphi_{\mathbf{S}})\in\mathbf{S}$. Choose an arbitrary $\mathbf{S}_i$.

By Lemma 4, for any $\mathbf{S}\in\mathbf{E}_c$, $(\Gamma,\varphi_{\mathbf{S}_i})\in\mathbf{S}$. It follows that $\Gamma\in(\bigcap_{\mathbf{S}\in\mathbf{E}_c}\mathbf{S})^\bullet$. □

6.2. Complete extensions

This subsection shows that for bisimilar frames, $(.)_\bullet$ and $(.)^\bullet$ are bijections when restricted to complete extensions.

The following proposition shows that $(.)_\bullet$ and $(.)^\bullet$ preserve complete extensions between bisimilar frames.

Proposition 3. *Let $\mathcal{F}^{\mathbb{D}}_\Sigma = (Ar_b(\Sigma), \mathcal{R}_{\mathbb{D}})$ be a BAF and $\mathbf{F}^{\mathbf{D}}_\Sigma = (Ar_p(\Sigma), \mathbf{R}_{\mathbf{D}})$ a PAF such that $\mathcal{F}^{\mathbb{D}}_\Sigma$ and $\mathbf{F}^{\mathbf{D}}_\Sigma$ are bisimilar.*

1. *If $\mathcal{S}$ is a complete extension in $\mathcal{F}^{\mathbb{D}}_\Sigma$, then $\mathcal{S}_\bullet$ is a complete extension in $\mathbf{F}^{\mathbf{D}}_\Sigma$.*
2. *If $\mathbf{S}$ is a complete extension in $\mathbf{F}^{\mathbf{D}}_\Sigma$, then $\mathbf{S}^\bullet$ is a complete extension in $\mathcal{F}^{\mathbb{D}}_\Sigma$.*

Proof. (1) Assume that $\mathcal{S}$ is a complete extension. To show that $\mathcal{S}_\bullet$ is a complete extension, we have to show that $\mathcal{S}_\bullet$ is conflict-free, defends each of its elements and contains each argument it defends.

- First we show that $\mathcal{S}_\bullet$ is conflict-free. We prove by contradiction and assume that there exist premise-conclusion arguments (Γ,φ) and $(\Delta,\psi)\in\mathcal{S}_\bullet$ such that (Γ,φ) attacks (Δ,ψ). Since $\mathcal{F}^{\mathbb{D}}_\Sigma$ and $\mathbf{F}^{\mathbf{D}}_\Sigma$ are bisimilar, Γ attacks Δ.
  Since $(\Gamma,\varphi),(\Delta,\psi)\in\mathcal{S}_\bullet$, $\Gamma,\Delta\in\mathcal{S}$, which contradicts the fact that $\mathcal{S}$ is conflict-free.
- Next we show that $\mathcal{S}_\bullet$ defends each of its elements. Let $(\Gamma,\varphi)\in\mathcal{S}_\bullet$ and assume that (Δ,ψ) attacks (Γ,φ). Since $\mathcal{F}^{\mathbb{D}}_\Sigma$ and $\mathbf{F}^{\mathbf{D}}_\Sigma$ are bisimilar, Δ attacks Γ.
  Since $(\Gamma,\varphi)\in\mathcal{S}_\bullet$, $\Gamma\in\mathcal{S}$. Since $\mathcal{S}$ is a complete extension, there exists $\Delta'\in\mathcal{S}$ such that Δ' attacks Δ.
  Since $\mathcal{F}^{\mathbb{D}}_\Sigma$ and $\mathbf{F}^{\mathbf{D}}_\Sigma$ are bisimilar, by Lemma 5, there exists a premise-conclusion argument (Δ'',φ) that attacks (Δ,ψ), where $\Delta''\subseteq\Delta'$.
  By Lemma 3, $(\Delta',\bigwedge\Delta')$ is a premise-conclusion argument. Since $\Delta'\in\mathcal{S}$, $(\Delta',\bigwedge\Delta')\in\mathcal{S}_\bullet$. Since $\Delta''\subseteq\Delta'$, by Lemma 4, $(\Delta'',\varphi)\in\mathcal{S}_\bullet$. It follows that $\mathcal{S}_\bullet$ attacks (Δ,ψ).
- Let (Γ,φ) be an argument defended by $\mathcal{S}_\bullet$. To show that $(\Gamma,\varphi)\in\mathcal{S}_\bullet$, it suffices to show that $\Gamma\in\mathcal{S}$. Since $\mathcal{S}$ is a complete extension, we show that $\Gamma\in\mathcal{S}$ by proving that $\mathcal{S}$ defends Γ. Let Δ be a base argument that attacks Γ.
  Since $\mathcal{F}^{\mathbb{D}}_\Sigma$ and $\mathbf{F}^{\mathbf{D}}_\Sigma$ are bisimilar, by Lemma 5, there exists a premise-conclusion argument (Δ',φ) that attacks (Γ,φ), where $\Delta'\subseteq\Delta$.
  Since (Γ,φ) is defended by $\mathcal{S}_\bullet$, there exists $(\Omega,\varphi')\in\mathcal{S}_\bullet$ that attacks (Δ',φ). Since $\mathcal{F}^{\mathbb{D}}_\Sigma$ and $\mathbf{F}^{\mathbf{D}}_\Sigma$ are bisimilar, Ω attacks Δ' and $\mathbb{D}$ is monotonic. Since $\Delta'\subseteq\Delta$, Ω attacks Δ.
  Since $(\Omega,\varphi')\in\mathcal{S}_\bullet$, $\Omega\in\mathcal{S}$. Therefore, $\mathcal{S}$ defends Γ.

(2) Assume that $\mathbf{S}$ is a complete extension. To show that $\mathbf{S}^\bullet$ is a complete extension, we have to show that $\mathbf{S}^\bullet$ is conflict-free, defends each of its elements and contains each argument it defends.

- We first show that $\mathbf{S}^\bullet$ is conflict-free. We prove by contradiction and assume that there exist $\Gamma,\Delta\in\mathbf{S}^\bullet$ such that Γ attacks Δ. Since $\Delta\in\mathbf{S}^\bullet$, there exists ψ such that $(\Delta,\psi)\in\mathbf{S}$.
  Since $\mathcal{F}^{\mathbb{D}}_\Sigma$ and $\mathbf{F}^{\mathbf{D}}_\Sigma$ are bisimilar, by Lemma 5, there exists a premise-conclusion argument (Γ',φ) that attacks (Δ,ψ), where $\Gamma'\subseteq\Gamma$.
  Since $\Gamma\in\mathbf{S}^\bullet$, there exists φ such that $(\Gamma,\varphi)\in\mathbf{S}$. By Lemma 4, $(\Gamma',\varphi)\in\mathbf{S}$.
  Since (Γ',φ) attacks (Δ,ψ) and they belong to $\mathbf{S}$, we have a contradiction against the fact that $\mathbf{S}$ is conflict-free.
- Then we show that $\mathbf{S}^\bullet$ defends each of its elements. Let $\Gamma\in\mathbf{S}^\bullet$ and assume that Δ attacks Γ. Since $\Gamma\in\mathbf{S}^\bullet$, there exists φ such that $(\Gamma,\varphi)\in\mathbf{S}$.

Since Δ attacks Γ and $\mathcal{F}_{\Sigma}^{\mathbb{D}}$ and $\mathbf{F}_{\Sigma}^{\mathbf{D}}$ are bisimilar, by Lemma 5, there exists a premise-conclusion argument (Δ', φ) that attacks (Γ, φ), where $\Delta' \subseteq \Delta$.

Since $\mathbf{S}$ is a complete extension, there exists $(\Omega, \varphi') \in \mathbf{S}$ that attacks (Δ', φ). Since $\mathcal{F}_{\Sigma}^{\mathbb{D}}$ and $\mathbf{F}_{\Sigma}^{\mathbf{D}}$ are bisimilar, Ω attacks Δ' and $\mathbb{D}$ is monotonic. Since $\Delta' \subseteq \Delta$, Ω attacks Δ.

Since $(\Omega, \varphi') \in \mathbf{S}$, $\Omega \in \mathbf{S}^{\bullet}$. Therefore, $\mathbf{S}^{\bullet}$ defends Γ.

– Let Γ be an argument defended by $\mathbf{S}^{\bullet}$. To show that $\Gamma \in \mathbf{S}^{\bullet}$, since $(\Gamma, \bigwedge \Gamma)$ is a premise-conclusion argument (Lemma 3), it suffices to show that $(\Gamma, \bigwedge \Gamma) \in \mathbf{S}$. Since $\mathbf{S}$ is a complete extension, it suffices to show that $\mathbf{S}$ defends $(\Gamma, \bigwedge \Gamma)$. Let (Δ, ψ) attack $(\Gamma, \bigwedge \Gamma)$. Since $\mathcal{F}_{\Sigma}^{\mathbb{D}}$ and $\mathbf{F}_{\Sigma}^{\mathbf{D}}$ are bisimilar, Δ attacks Γ. Since $\mathbf{S}^{\bullet}$ defends Γ, there exists $\Omega \in \mathbf{S}^{\bullet}$ that attacks Δ.

Since $\mathcal{F}_{\Sigma}^{\mathbb{D}}$ and $\mathbf{F}_{\Sigma}^{\mathbf{D}}$ are bisimilar, by Lemma 5, there exists a premise-conclusion argument (Ω', φ) that attacks (Δ, ψ), where $\Omega' \subseteq \Omega$.

Since $\Omega \in \mathbf{S}^{\bullet}$, there exists φ such that $(\Omega, \varphi) \in \mathbf{S}$. Since $\Omega' \subseteq \Omega$, by Lemma 4, $(\Omega', \varphi) \in \mathbf{S}$. Therefore, $\mathbf{S}$ defends $(\Gamma, \bigwedge \Gamma)$. □

Recall that $\mathcal{E}_{co}(\mathcal{F}_{\Sigma}^{\mathbb{D}})$ is the set of complete extensions in $\mathcal{F}_{\Sigma}^{\mathbb{D}}$ and that $\mathbf{E}_{co}(\mathbf{F}_{\Sigma}^{\mathbf{D}})$ is the set of complete extensions in $\mathbf{F}_{\Sigma}^{\mathbf{D}}$.

The following proposition shows that for bisimilar frames, $(.)_{\bullet}$ and $(.)^{\bullet}$ are bijections when restricted to complete extensions.

Proposition 4. *Let $\mathcal{F}_{\Sigma}^{\mathbb{D}} = (Ar_b(\Sigma), \mathcal{R}_{\mathbb{D}})$ be a BAF and $\mathbf{F}_{\Sigma}^{\mathbf{D}} = (Ar_p(\Sigma), \mathbf{R}_{\mathbf{D}})$ a PAF such that $\mathcal{F}_{\Sigma}^{\mathbb{D}}$ and $\mathbf{F}_{\Sigma}^{\mathbf{D}}$ are bisimilar.*

1. *$(.)_{\bullet} : \mathcal{E}_{co}(\mathcal{F}_{\Sigma}^{\mathbb{D}}) \to \mathbf{E}_{co}(\mathbf{F}_{\Sigma}^{\mathbf{D}})$ is a bijection.*
2. *$(.)^{\bullet} : \mathbf{E}_{co}(\mathbf{F}_{\Sigma}^{\mathbf{D}}) \to \mathcal{E}_{co}(\mathcal{F}_{\Sigma}^{\mathbb{D}})$ is a bijection.*

Proof. (1) By Proposition 3, $(.)_{\bullet} : \mathcal{E}_{co}(\mathcal{F}_{\Sigma}^{\mathbb{D}}) \to \mathbf{E}_{co}(\mathbf{F}_{\Sigma}^{\mathbf{D}})$ is well-defined as a function. We first show that $(.)_{\bullet}$ is an injection. Let $\mathcal{S}, \mathcal{S}' \in \mathcal{E}_{co}(\mathcal{F}_{\Sigma}^{\mathbb{D}})$ such that $\mathcal{S} \neq \mathcal{S}'$. Without loss of generality, assume that $\Gamma \in \mathcal{S}$ and $\Gamma \notin \mathcal{S}'$. By Lemma 3, $(\Gamma, \bigwedge \Gamma)$ is a premise-conclusion argument. By definition, $(\Gamma, \bigwedge \Gamma) \in \mathcal{S}_{\bullet}$ and $(\Gamma, \bigwedge \Gamma) \notin \mathcal{S}'_{\bullet}$. Therefore, $\mathcal{S}_{\bullet} \neq \mathcal{S}'_{\bullet}$.

Now we show that $(.)_{\bullet}$ is a surjection. Let $\mathbf{S} \in \mathbf{E}_{co}(\mathbf{F}_{\Sigma}^{\mathbf{D}})$. By Lemma 6, $(\mathbf{S}^{\bullet})_{\bullet} = \mathbf{S}$. Therefore, $(.)_{\bullet}$ is a surjection.

(2) By Proposition 3, $(.)^{\bullet} : \mathbf{E}_{co}(\mathbf{F}_{\Sigma}^{\mathbf{D}}) \to \mathcal{E}_{co}(\mathcal{F}_{\Sigma}^{\mathbb{D}})$ is well-defined as a function. We first show that $(.)^{\bullet}$ is an injection. Let $\mathbf{S}, \mathbf{S}' \in \mathbf{E}_{co}(\mathbf{F}_{\Sigma}^{\mathbf{D}})$ such that $\mathbf{S} \neq \mathbf{S}'$. Without loss of generality, assume that $(\Gamma, \varphi) \in \mathbf{S}$ and $(\Gamma, \varphi) \notin \mathbf{S}'$. It follows that $\Gamma \in \mathbf{S}^{\bullet}$. Now we show that $\Gamma \notin (\mathbf{S}')^{\bullet}$. We prove by contradiction and assume that $\Gamma \in (\mathbf{S}')^{\bullet}$. Then there exist ψ such that $(\Gamma, \psi) \in \mathbf{S}'$. Since $(\Gamma, \psi) \in \mathbf{S}'$, by Lemma 4, $(\Gamma, \varphi) \in \mathbf{S}'$, contradiction.

Now we show that $(.)^{\bullet}$ is a surjection. By Lemma 6, $(\mathcal{S}_{\bullet})^{\bullet} = \mathcal{S}$. Therefore, $(.)^{\bullet}$ is a surjection. □

6.3. Stable, grounded, preferred and semi-stable extensions

This subsection shows that for bisimilar frames, $(.)_{\bullet}$ and $(.)^{\bullet}$ are bijections when restricted to stable/grounded/preferred/semi-stable extensions.

The following proposition shows that $(.)_{\bullet}$ and $(.)^{\bullet}$ preserve stable, grounded, preferred and semi-stable extensions between bisimilar frames.

Proposition 5. *Let $\mathcal{F}_{\Sigma}^{\mathbb{D}} = (Ar_b(\Sigma), \mathcal{R}_{\mathbb{D}})$ be a BAF and $\mathbf{F}_{\Sigma}^{\mathbf{D}} = (Ar_p(\Sigma), \mathbf{R}_{\mathbf{D}})$ a PAF such that $\mathcal{F}_{\Sigma}^{\mathbb{D}}$ and $\mathbf{F}_{\Sigma}^{\mathbf{D}}$ are bisimilar.*

1. *If $\mathcal{S}$ is a stable/grounded/preferred/semi-stable extension in $\mathcal{F}_{\Sigma}^{\mathbb{D}}$, then $\mathcal{S}_{\bullet}$ is a stable/grounded/preferred/semi-stable extension in $\mathbf{F}_{\Sigma}^{\mathbf{D}}$.*
2. *If $\mathbf{S}$ is a stable/grounded/preferred/semi-stable extension in $\mathbf{F}_{\Sigma}^{\mathbf{D}}$, then $\mathbf{S}^{\bullet}$ is a stable/grounded/preferred/semi-stable extension in $\mathcal{F}_{\Sigma}^{\mathbb{D}}$.*

Proof. (1) Assume that $\mathcal{S}$ is stable. To show that $\mathcal{S}_{\bullet}$ is stable, by Proposition 3, it suffices to show that $\mathcal{S}_{\bullet}$ attacks each argument that is not in $\mathcal{S}_{\bullet}$. Let (Γ, φ) be a premise-conclusion argument that does not belong to $\mathcal{S}_{\bullet}$. It follows that $\Gamma \notin \mathcal{S}$. Since $\mathcal{S}$ is stable, there exists a base argument $\Delta \in \mathcal{S}$ that attacks Γ.

Since $\mathcal{F}_{\Sigma}^{\mathbb{D}}$ and $\mathbf{F}_{\Sigma}^{\mathbf{D}}$ are bisimilar, by Lemma 5, there exists a premise-conclusion argument (Δ', φ) that attacks (Γ, φ), where $\Delta' \subseteq \Delta$.

By Lemma 3, $(\Delta, \bigwedge \Delta)$ is a premise-conclusion argument. Since $\Delta \in \mathcal{S}$, $(\Delta, \bigwedge \Delta) \in \mathcal{S}_{\bullet}$. Since $\Delta' \subseteq \Delta$, by Lemma 4, $(\Delta', \varphi) \in \mathcal{S}_{\bullet}$. Therefore, $\mathcal{S}_{\bullet}$ attacks (Γ, φ).

Assume that $\mathcal{S}$ is grounded. By Proposition 3, $\mathcal{S}_{\bullet}$ is complete. To show that $\mathcal{S}_{\bullet}$ is grounded, we have to show that $\mathcal{S}_{\bullet}$ is the smallest complete extension in $\mathbf{F}_{\Sigma}^{\mathbf{D}}$. Let $\mathbf{S}'$ be a complete extension in $\mathbf{F}_{\Sigma}^{\mathbf{D}}$. By Proposition 3, $(\mathbf{S}')^{\bullet}$ is a complete extension in $\mathcal{F}_{\Sigma}^{\mathbb{D}}$. Since $\mathcal{S}$ is grounded in $\mathcal{F}_{\Sigma}^{\mathbb{D}}$, $\mathcal{S} \subseteq (\mathbf{S}')^{\bullet}$. By Lemma 6, $\mathcal{S}_{\bullet} \subseteq ((\mathbf{S}')^{\bullet})_{\bullet}$ and $((\mathbf{S}')^{\bullet})_{\bullet} = \mathbf{S}'$. Therefore, $\mathcal{S}_{\bullet} \subseteq \mathbf{S}'$.

Assume that $\mathcal{S}$ is preferred. By Proposition 3, $\mathcal{S}_{\bullet}$ is complete. To show that $\mathcal{S}_{\bullet}$ is preferred, we need to show that $\mathcal{S}_{\bullet}$ is a maximal complete extension. Let $\mathbf{S}'$ be a complete extension in $\mathbf{F}_{\Sigma}^{\mathbf{D}}$ such that $\mathcal{S}_{\bullet} \subseteq \mathbf{S}'$. We need to show that $\mathcal{S}_{\bullet} = \mathbf{S}'$. Since $\mathbf{S}'$ is a complete extension in $\mathbf{F}_{\Sigma}^{\mathbf{D}}$, by Proposition 3, $(\mathbf{S}')^{\bullet}$ is a complete extension in $\mathcal{F}_{\Sigma}^{\mathbb{D}}$.

Since $\mathcal{S}_{\bullet} \subseteq \mathbf{S}'$, by Lemma 6, $(\mathcal{S}_{\bullet})^{\bullet} \subseteq (\mathbf{S}')^{\bullet}$. By Lemma 6 again, $(\mathcal{S}_{\bullet})^{\bullet} = \mathcal{S}$. It follows that $\mathcal{S} \subseteq (\mathbf{S}')^{\bullet}$. Since $\mathcal{S}$ is preferred and $(\mathbf{S}')^{\bullet}$ is complete, $\mathcal{S} = (\mathbf{S}')^{\bullet}$. By Lemma 6, $\mathcal{S}_{\bullet} = ((\mathbf{S}')^{\bullet})_{\bullet}$ and $((\mathbf{S}')^{\bullet})_{\bullet} = \mathbf{S}'$. It follows that $\mathcal{S}_{\bullet} = \mathbf{S}'$.

Assume that $\mathcal{S}$ is semi-stable. By Proposition 3, $\mathcal{S}_{\bullet}$ is complete. To show that $\mathcal{S}_{\bullet}$ is semi-stable, we need to show that $\mathcal{S}_{\bullet} \cup (\mathcal{S}_{\bullet})^{+}$ is maximal among complete extensions. Let $\mathbf{S}$ be a complete extension in $\mathbf{F}_{\Sigma}^{\mathbf{D}}$ such that $\mathcal{S}_{\bullet} \cup (\mathcal{S}_{\bullet})^{+} \subseteq \mathbf{S} \cup \mathbf{S}^{+}$. We need to show that $\mathbf{S} \cup \mathbf{S}^{+} = \mathcal{S}_{\bullet} \cup (\mathcal{S}_{\bullet})^{+}$.

Next we use Lemma 6 to transform $\mathcal{S}_{\bullet} \cup (\mathcal{S}_{\bullet})^{+} \subseteq \mathbf{S} \cup \mathbf{S}^{+}$:

$$(\mathcal{S}_{\bullet} \cup (\mathcal{S}_{\bullet})^{+})^{\bullet} \subseteq (\mathbf{S} \cup \mathbf{S}^{+})^{\bullet} \qquad \text{Item (2)}$$
$$(\mathcal{S}_{\bullet})^{\bullet} \cup ((\mathcal{S}_{\bullet})^{+})^{\bullet} \subseteq \mathbf{S}^{\bullet} \cup (\mathbf{S}^{+})^{\bullet} \qquad \text{Item (6)}$$
$$(\mathcal{S}_{\bullet})^{\bullet} \cup ((\mathcal{S}_{\bullet})^{\bullet})^{+} \subseteq \mathbf{S}^{\bullet} \cup (\mathbf{S}^{\bullet})^{+} \qquad \text{Item (8)}$$
$$\mathcal{S} \cup \mathcal{S}^{+} \subseteq \mathbf{S}^{\bullet} \cup (\mathbf{S}^{\bullet})^{+} \qquad \text{Item (3)}$$

Since $\mathcal{S}$ is semi-stable, $\mathcal{S} \cup \mathcal{S}^{+} = \mathbf{S}^{\bullet} \cup (\mathbf{S}^{\bullet})^{+}$. Use Lemma 6 to transform this equation:

$$(\mathcal{S} \cup \mathcal{S}^{+})_{\bullet} = (\mathbf{S}^{\bullet} \cup (\mathbf{S}^{\bullet})^{+})_{\bullet} \qquad \text{Item (1)}$$
$$\mathcal{S}_{\bullet} \cup (\mathcal{S}^{+})_{\bullet} = (\mathbf{S}^{\bullet})_{\bullet} \cup ((\mathbf{S}^{\bullet})^{+})_{\bullet} \qquad \text{Item (5)}$$
$$\mathcal{S}_{\bullet} \cup (\mathcal{S}_{\bullet})^{+} = (\mathbf{S}^{\bullet})_{\bullet} \cup ((\mathbf{S}^{\bullet})_{\bullet})^{+} \qquad \text{Item (7)}$$
$$\mathcal{S}_{\bullet} \cup (\mathcal{S}_{\bullet})^{+} = \mathbf{S} \cup (\mathbf{S})^{+} \qquad \text{Item (4)}$$

(2) Assume that $\mathbf{S}$ is stable. To show that $\mathbf{S}^{\bullet}$ is stable, it suffices to show that $\mathbf{S}^{\bullet}$ attacks each argument that is not in it. Let Γ be a base argument that is not in $\mathbf{S}^{\bullet}$. By Lemma 3, $(\Gamma, \bigwedge \Gamma)$ is a premise-conclusion argument. It follows that $(\Gamma, \bigwedge \Gamma) \notin \mathbf{S}$. Then there exists a premise-conclusion argument $(\Delta, \psi) \in \mathbf{S}$ that attacks $(\Gamma, \bigwedge \Gamma)$. Since $\mathcal{F}^{\mathbb{D}}_{\Sigma}$ and $\mathbf{F}^{\mathbf{D}}_{\Sigma}$ are bisimilar, Δ attacks Γ. Since $(\Delta, \psi) \in \mathbf{S}$, $\Delta \in \mathbf{S}^{\bullet}$. Therefore, $\mathbf{S}^{\bullet}$ attacks Γ.

Assume that $\mathbf{S}$ is grounded. By Proposition 3, $\mathbf{S}^{\bullet}$ is complete. To show that $\mathbf{S}^{\bullet}$ is grounded, it suffices to show that $\mathbf{S}^{\bullet}$ is the smallest complete extension in $\mathcal{F}^{\mathbb{D}}_{\Sigma}$. Let $\mathcal{S}'$ be a complete extension in $\mathcal{F}^{\mathbb{D}}_{\Sigma}$. By Proposition 3, $(\mathcal{S}')_{\bullet}$ is a complete extension in $\mathbf{F}^{\mathbf{D}}_{\Sigma}$. Since $\mathbf{S}$ is the grounded extension in $\mathbf{F}^{\mathbf{D}}_{\Sigma}$, $\mathbf{S} \subseteq (\mathcal{S}')_{\bullet}$. By Lemma 6, $\mathbf{S}^{\bullet} \subseteq ((\mathcal{S}')_{\bullet})^{\bullet}$ and $((\mathcal{S}')_{\bullet})^{\bullet} = \mathcal{S}'$. Therefore, $\mathbf{S}^{\bullet} \subseteq \mathcal{S}'$.

Assume that $\mathbf{S}$ is preferred. By Proposition 3, $\mathbf{S}^{\bullet}$ is complete. To show that $\mathbf{S}^{\bullet}$ is preferred, we need to show that $\mathbf{S}^{\bullet}$ is a maximal complete extension. Let $\mathcal{S}'$ be a complete extension in $\mathcal{F}^{\mathbb{D}}_{\Sigma}$ such that $\mathbf{S}^{\bullet} \subseteq \mathcal{S}'$. We need to show that $\mathbf{S}^{\bullet} = \mathcal{S}'$. Since $\mathcal{S}'$ is a complete extension in $\mathcal{F}^{\mathbb{D}}_{\Sigma}$, by Proposition 3, $(\mathcal{S}')_{\bullet}$ is a complete extension in $\mathbf{F}^{\mathbf{D}}_{\Sigma}$. Since $\mathbf{S}^{\bullet} \subseteq \mathcal{S}'$, by Lemma 6, $(\mathbf{S}^{\bullet})_{\bullet} \subseteq (\mathcal{S}')_{\bullet}$. By Lemma 6 again, $(\mathbf{S}^{\bullet})_{\bullet} = \mathbf{S}$. It follows that $\mathbf{S} \subseteq (\mathcal{S}')_{\bullet}$. Since $\mathbf{S}$ is preferred and $(\mathcal{S}')_{\bullet}$ is complete, $\mathbf{S} = (\mathcal{S}')_{\bullet}$. By Lemma 6, $\mathbf{S}^{\bullet} = ((\mathcal{S}')_{\bullet})^{\bullet}$ and $((\mathcal{S}')_{\bullet})^{\bullet} = \mathbf{S}'$. It follows that $\mathbf{S}^{\bullet} = \mathcal{S}'$.

Assume that $\mathbf{S}$ is semi-stable. By Proposition 3, $\mathbf{S}^{\bullet}$ is complete. To show that $\mathbf{S}^{\bullet}$ is semi-stable, we need to show that $\mathbf{S}^{\bullet} \cup (\mathbf{S}^{\bullet})^{+}$ is maximal among complete extensions. Let $\mathcal{S}$ be a complete extension in $\mathcal{F}^{\mathbb{D}}_{\Sigma}$ such that $\mathbf{S}^{\bullet} \cup (\mathbf{S}^{\bullet})^{+} \subseteq \mathcal{S} \cup \mathcal{S}^{+}$. We need to show that $\mathbf{S}^{\bullet} \cup (\mathbf{S}^{\bullet})^{+} = \mathcal{S} \cup \mathcal{S}^{+}$.

We use Lemma 6 to transform $\mathbf{S}^{\bullet} \cup (\mathbf{S}^{\bullet})^{+} \subseteq \mathcal{S} \cup \mathcal{S}^{+}$:

$$(\mathbf{S}^{\bullet} \cup (\mathbf{S}^{\bullet})^{+})_{\bullet} \subseteq (\mathcal{S} \cup \mathcal{S}^{+})_{\bullet} \qquad \text{Item (1)}$$
$$(\mathbf{S}^{\bullet})_{\bullet} \cup ((\mathbf{S}^{\bullet})^{+})_{\bullet} \subseteq \mathcal{S}_{\bullet} \cup (\mathcal{S}^{+})_{\bullet} \qquad \text{Item (5)}$$
$$(\mathbf{S}^{\bullet})_{\bullet} \cup ((\mathbf{S}^{\bullet})_{\bullet})^{+} \subseteq \mathcal{S}_{\bullet} \cup (\mathcal{S}_{\bullet})^{+} \qquad \text{Item (7)}$$
$$\mathbf{S} \cup \mathbf{S}^{+} \subseteq \mathcal{S}_{\bullet} \cup (\mathcal{S}_{\bullet})^{+} \qquad \text{Item (4)}$$

Since $\mathbf{S}$ is semi-stable, $\mathbf{S} \cup \mathbf{S}^{+} = \mathcal{S}_{\bullet} \cup (\mathcal{S}_{\bullet})^{+}$. Use Lemma 6 to transform this equation:

$$(\mathbf{S} \cup \mathbf{S}^{+})^{\bullet} = (\mathcal{S}_{\bullet} \cup (\mathcal{S}_{\bullet})^{+})^{\bullet} \qquad \text{Item (2)}$$
$$\mathbf{S}^{\bullet} \cup (\mathbf{S}^{+})^{\bullet} = (\mathcal{S}_{\bullet})^{\bullet} \cup ((\mathcal{S}_{\bullet})^{+})^{\bullet} \qquad \text{Item (6)}$$
$$\mathbf{S}^{\bullet} \cup (\mathbf{S}^{\bullet})^{+} = (\mathcal{S}_{\bullet})^{\bullet} \cup ((\mathcal{S}_{\bullet})^{\bullet})^{+} \qquad \text{Item (8)}$$
$$\mathbf{S}^{\bullet} \cup (\mathbf{S}^{\bullet})^{+} = \mathcal{S} \cup (\mathcal{S})^{+} \qquad \text{Item (3)} \quad \square$$

The following proposition shows that for bisimilar frames, $(.)_{\bullet}$ and $(.)^{\bullet}$ are bijections when restricted to stable/grounded/preferred/semi-stable extensions.

Proposition 6. *Let $\mathcal{F}^{\mathbb{D}}_{\Sigma} = (Ar_b(\Sigma), \mathcal{R}_{\mathbb{D}})$ be a BAF and $\mathbf{F}^{\mathbf{D}}_{\Sigma} = (Ar_p(\Sigma), \mathbf{R}_{\mathbf{D}})$ a PAF such that $\mathcal{F}^{\mathbb{D}}_{\Sigma}$ and $\mathbf{F}^{\mathbf{D}}_{\Sigma}$ are bisimilar. For $X \in \{st, gr, pr, ss\}$,*

1. *$(.)_{\bullet} : \mathcal{E}_X(\mathcal{F}^{\mathbb{D}}_{\Sigma}) \to \mathbf{E}_X(\mathbf{F}^{\mathbf{D}}_{\Sigma})$ is a bijection, and*
2. *$(.)^{\bullet} : \mathbf{E}_X(\mathbf{F}^{\mathbf{D}}_{\Sigma}) \to \mathcal{E}_X(\mathcal{F}^{\mathbb{D}}_{\Sigma})$ is a bijection.*

Proof. By Proposition 5, $(.)_{\bullet} : \mathcal{E}_X(\mathcal{F}^{\mathbb{D}}_{\Sigma}) \to \mathbf{E}_X(\mathbf{F}^{\mathbf{D}}_{\Sigma})$ and $(.)^{\bullet} : \mathbf{E}_X(\mathbf{F}^{\mathbf{D}}_{\Sigma}) \to \mathcal{E}_X(\mathcal{F}^{\mathbb{D}}_{\Sigma})$ are well-defined. Since each stable/grounded/preferred/semi-stable extension is a complete extension, by Proposition 4, $(.)_{\bullet} : \mathcal{E}_X(\mathcal{F}^{\mathbb{D}}_{\Sigma}) \to \mathbf{E}_X(\mathbf{F}^{\mathbf{D}}_{\Sigma})$ and $(.)^{\bullet} : \mathbf{E}_X(\mathbf{F}^{\mathbf{D}}_{\Sigma}) \to \mathcal{E}_X(\mathcal{F}^{\mathbb{D}}_{\Sigma})$ are injections. By Lemma 6, $(\mathbf{S}^{\bullet})_{\bullet} = \mathbf{S}$ and $(\mathcal{S}_{\bullet})^{\bullet} = \mathcal{S}$. Therefore, $(.)_{\bullet} : \mathcal{E}_X(\mathcal{F}^{\mathbb{D}}_{\Sigma}) \to \mathbf{E}_X(\mathbf{F}^{\mathbf{D}}_{\Sigma})$ and $(.)^{\bullet} : \mathbf{E}_X(\mathbf{F}^{\mathbf{D}}_{\Sigma}) \to \mathcal{E}_X(\mathcal{F}^{\mathbb{D}}_{\Sigma})$ are bijections. $\square$

6.4. Ideal and eager extensions

Before showing that for bisimilar frames, $(.)_\bullet$ and $(.)^\bullet$ are bijections when restricted to ideal/eager extensions, we need the following lemma.

Lemma 7. *Let $\mathcal{F}^{\mathbb{D}}_{\Sigma} = (Ar_b(\Sigma), \mathcal{R}_{\mathbb{D}})$ be a BAF and $\mathbf{F}^{\mathbf{D}}_{\Sigma} = (Ar_p(\Sigma), \mathbf{R}_{\mathbf{D}})$ a PAF such that $\mathcal{F}^{\mathbb{D}}_{\Sigma}$ and $\mathbf{F}^{\mathbf{D}}_{\Sigma}$ are bisimilar.*

1. *If $\mathcal{S}$ is a complete extension that is contained in each preferred/semi-stable extension in $\mathcal{F}^{\mathbb{D}}_{\Sigma}$, then $\mathcal{S}_\bullet$ is a complete extension in $\mathbf{F}^{\mathbf{D}}_{\Sigma}$ that is contained in each preferred/semi-stable extension.*
2. *If $\mathbf{S}$ is a complete extension that is contained in each preferred/semi-stable extension in $\mathcal{F}^{\mathbb{D}}_{\Sigma}$, then $\mathbf{S}^\bullet$ is a complete extension in $\mathcal{F}^{\mathbb{D}}_{\Sigma}$ that is contained in each preferred/semi-stable extension.*

Proof. (1) Since $\mathcal{S}$ is complete, by Proposition 3, $\mathcal{S}_\bullet$ is complete. Let $\mathcal{E}$ be the set of preferred/semi-stable extensions in $\mathcal{F}^{\mathbb{D}}_{\Sigma}$. Since $\mathcal{S}$ is contained in each preferred/semi-stable extension in $\mathcal{F}^{\mathbb{D}}_{\Sigma}$, $\mathcal{S} \subseteq \bigcap_{\mathcal{S}' \in \mathcal{E}} \mathcal{S}'$. We use Lemma 6 to transform $\mathcal{S} \subseteq \bigcap_{\mathcal{S}' \in \mathcal{E}} \mathcal{S}'$:

$$\mathcal{S}_\bullet \subseteq (\bigcap_{\mathcal{S}' \in \mathcal{E}} \mathcal{S}')_\bullet \qquad \text{Item (1)}$$

$$\mathcal{S}_\bullet \subseteq \bigcap_{\mathcal{S}' \in \mathcal{E}} \mathcal{S}'_\bullet \qquad \text{Item (9)}$$

By Proposition 6, $(.)_\bullet : \mathcal{E}_{pr}(\mathcal{F}^{\mathbb{D}}_{\Sigma}) \to \mathbf{E}_{pr}(\mathbf{F}^{\mathbf{D}}_{\Sigma})$ and $(.)_\bullet : \mathcal{E}_{ss}(\mathcal{F}^{\mathbb{D}}_{\Sigma}) \to \mathbf{E}_{ss}(\mathbf{F}^{\mathbf{D}}_{\Sigma})$ are bijections. It follows that $\{\mathcal{S}'_\bullet \mid \mathcal{S}' \in \mathcal{E}\}$ is the set of preferred/semi-stable extensions in $\mathbf{F}^{\mathbf{D}}_{\Sigma}$. Since $\mathcal{S}_\bullet \subseteq \bigcap_{\mathcal{S}' \in \mathcal{E}} \mathcal{S}'_\bullet$, $\mathcal{S}_\bullet$ is contained in each preferred/semi-stable extension in $\mathbf{F}^{\mathbf{D}}_{\Sigma}$.

(2) Since $\mathbf{S}$ is complete, by Proposition 3, $\mathbf{S}^\bullet$ is complete. Let $\mathbf{E}$ be the set of preferred/ semi-stable extensions in $\mathbf{F}^{\mathbf{D}}_{\Sigma}$. Since $\mathbf{S}$ is contained in each preferred/semi-stable extension in $\mathbf{F}^{\mathbf{D}}_{\Sigma}$, $\mathbf{S} \subseteq \bigcap_{\mathbf{S}' \in \mathbf{E}} \mathbf{S}'$. We use Lemma 6 to transform $\mathbf{S} \subseteq \bigcap_{\mathbf{S}' \in \mathbf{E}} \mathbf{S}'$:

$$\mathbf{S}^\bullet \subseteq (\bigcap_{\mathbf{S}' \in \mathbf{E}} \mathbf{S}')^\bullet \qquad \text{Item (2)}$$

$$\mathbf{S}^\bullet \subseteq \bigcap_{\mathbf{S}' \in \mathbf{E}} \mathbf{S}'^\bullet \qquad \text{Item (3)}$$

By Proposition 6, $(.)^\bullet : \mathbf{E}_{pr}(\mathbf{F}^{\mathbf{D}}_{\Sigma}) \to \mathcal{E}_{pr}(\mathcal{F}^{\mathbb{D}}_{\Sigma})$ and $(.)^\bullet : \mathbf{E}_{ss}(\mathbf{F}^{\mathbf{D}}_{\Sigma}) \to \mathcal{E}_{ss}(\mathcal{F}^{\mathbb{D}}_{\Sigma})$ are bijections. It follows that $\{\mathbf{S}'_\bullet \mid \mathbf{S}' \in \mathbf{E}\}$ is the set of preferred/semi-stable extensions in $\mathcal{F}^{\mathbb{D}}_{\Sigma}$. Since $\mathbf{S}^\bullet \subseteq \bigcap_{\mathbf{S}' \in \mathbf{E}} \mathbf{S}'^\bullet$, $\mathbf{S}^\bullet$ is contained in each preferred/semi-stable extension in $\mathcal{F}^{\mathbb{D}}_{\Sigma}$. □

The following proposition shows that $(.)_\bullet$ and $(.)^\bullet$ preserve ideal and eager extensions between bisimilar frames.

Proposition 7. *Let $\mathcal{F}^{\mathbb{D}}_{\Sigma} = (Ar_b(\Sigma), \mathcal{R}_{\mathbb{D}})$ be a BAF and $\mathbf{F}^{\mathbf{D}}_{\Sigma} = (Ar_p(\Sigma), \mathbf{R}_{\mathbf{D}})$ a PAF such that $\mathcal{F}^{\mathbb{D}}_{\Sigma}$ and $\mathbf{F}^{\mathbf{D}}_{\Sigma}$ are bisimilar.*

1. *If $\mathcal{S}$ is an ideal/eager extension in $\mathcal{F}^{\mathbb{D}}_{\Sigma}$, then $\mathcal{S}_\bullet$ is an ideal/eager extension in $\mathbf{F}^{\mathbf{D}}_{\Sigma}$.*
2. *If $\mathbf{S}$ is an ideal/eager extension in $\mathbf{F}^{\mathbf{D}}_{\Sigma}$, then $\mathbf{S}^\bullet$ is an ideal/eager extension in $\mathcal{F}^{\mathbb{D}}_{\Sigma}$.*

Proof. (1) Assume that $\mathcal{S}$ is an ideal/eager extension. To show that $\mathcal{S}_\bullet$ is ideal/eager, we need to show that $\mathcal{S}_\bullet$ is a maximal complete extension in $\mathbf{F}^{\mathbf{D}}_{\Sigma}$ that is contained in each preferred/semi-stable extension. By Proposition 3, $\mathcal{S}_\bullet$ is complete.

- Since $\mathcal{S}$ is ideal/eager, $\mathcal{S}$ is contained in each preferred/semi-stable extension in $\mathcal{F}^{\mathbb{D}}_{\Sigma}$. By Lemma 7, $\mathcal{S}_\bullet$ is contained in each preferred/semi-stable extension in $\mathbf{F}^{\mathbf{D}}_{\Sigma}$.
- Now we show that $\mathcal{S}_\bullet$ is a maximal complete extension that is contained in each preferred/semi-stable extension. Let $\mathbf{S}$ be a complete extension in $\mathbf{F}^{\mathbf{D}}_{\Sigma}$ that is contained in each preferred/semi-stable extension such that $\mathcal{S}_\bullet \subseteq \mathbf{S}$. We need to show that $\mathcal{S}_\bullet = \mathbf{S}$. We use Lemma 6 to transform $\mathcal{S}_\bullet \subseteq \mathbf{S}$:

  $$(\mathcal{S}_\bullet)^\bullet \subseteq \mathbf{S}^\bullet \qquad \text{Item (2)}$$

  $$\mathcal{S} \subseteq \mathbf{S}^\bullet \qquad \text{Item (3)}$$

  Since $\mathbf{S}$ is a complete extension in $\mathbf{F}^{\mathbf{D}}_{\Sigma}$ that is contained in each preferred/semi-stable extension, by Lemma 7, $\mathbf{S}^\bullet$ is contained in each preferred/semi-stable extension in $\mathcal{F}^{\mathbb{D}}_{\Sigma}$. Since $\mathcal{S}$ is ideal/eager and $\mathcal{S} \subseteq \mathbf{S}^\bullet$, we have $\mathcal{S} = \mathbf{S}^\bullet$. Use Lemma 6 to transform $\mathcal{S} = \mathbf{S}^\bullet$:

  $$\mathcal{S}_\bullet = (\mathbf{S}^\bullet)_\bullet \qquad \text{Item (1)}$$

  $$\mathcal{S}_\bullet = \mathbf{S} \qquad \text{Item (4)}$$

(2) Assume that $\mathbf{S}$ is an ideal/eager extension. To show that $\mathbf{S}^\bullet$ is ideal/eager, we need to show that $\mathbf{S}^\bullet$ is a maximal complete extension in $\mathcal{F}^{\mathbb{D}}_{\Sigma}$ that is contained in each preferred/semi-stable extension. By Proposition 3, $\mathbf{S}^\bullet$ is complete.

– Since $\mathbf{S}$ is ideal/eager, $\mathbf{S}$ is contained in each preferred/semi-table extension in $\mathbf{F}^{\mathbf{D}}_{\Sigma}$. By Proposition 7, $\mathbf{S}^{\bullet}$ is contained in each preferred/semi-stable extension in $\mathcal{F}^{\mathbb{D}}_{\Sigma}$.
– Now we show that $\mathbf{S}^{\bullet}$ is a maximal complete extension in $\mathcal{F}^{\mathbb{D}}_{\Sigma}$ that is contained in each preferred/semi-stable extension. Let $\mathcal{S}$ be a complete extension in $\mathcal{F}^{\mathbb{D}}_{\Sigma}$ that is contained in each preferred/semi-stable extension such that $\mathbf{S}^{\bullet} \subseteq \mathcal{S}$. We need to show that $\mathbf{S}^{\bullet} = \mathcal{S}$. Use Lemma 6 to transform $\mathbf{S}^{\bullet} \subseteq \mathcal{S}$:

$$(\mathbf{S}^{\bullet})_{\bullet} \subseteq \mathcal{S}_{\bullet} \qquad \text{Item (1)}$$
$$\mathbf{S} \subseteq \mathcal{S}_{\bullet} \qquad \text{Item (4)}$$

Since $\mathcal{S}$ is a complete extension in $\mathcal{F}^{\mathbb{D}}_{\Sigma}$ that is contained in each preferred/semi-stable extension, by Lemma 7, $\mathcal{S}_{\bullet}$ is contained in each preferred/semistable extension in $\mathbf{F}^{\mathbf{D}}_{\Sigma}$. Since $\mathbf{S}$ is ideal/eager and $\mathbf{S} \subseteq \mathcal{S}_{\bullet}$, we have $\mathbf{S} = \mathcal{S}_{\bullet}$. Use Lemma 6 to transform $\mathbf{S} = \mathcal{S}_{\bullet}$:

$$\mathbf{S}^{\bullet} = (\mathcal{S}_{\bullet})^{\bullet} \qquad \text{Item (2)}$$
$$\mathbf{S}^{\bullet} = \mathcal{S} \qquad \text{Item (3)} \quad \square$$

The following proposition shows that for bisimilar frames, $(.)_{\bullet}$ and $(.)^{\bullet}$ are bijections when restricted to ideal/eager extensions.

Proposition 8. *Let $\mathcal{F}^{\mathbb{D}}_{\Sigma} = (Ar_b(\Sigma), \mathcal{R}_{\mathbb{D}})$ be a BAF and $\mathbf{F}^{\mathbf{D}}_{\Sigma} = (Ar_p(\Sigma), \mathbf{R}_{\mathbf{D}})$ a PAF such that $\mathcal{F}^{\mathbb{D}}_{\Sigma}$ and $\mathbf{F}^{\mathbf{D}}_{\Sigma}$ are bisimilar. For $X \in \{id, ea\}$,*

1. $(.)_{\bullet} : \mathcal{E}_X(\mathcal{F}^{\mathbb{D}}_{\Sigma}) \to \mathbf{E}_X(\mathbf{F}^{\mathbf{D}}_{\Sigma})$ is a bijection, and
2. $(.)^{\bullet} : \mathbf{E}_X(\mathbf{F}^{\mathbf{D}}_{\Sigma}) \to \mathcal{E}_X(\mathcal{F}^{\mathbb{D}}_{\Sigma})$ is a bijection.

Proof. By Proposition 7, $(.)_{\bullet} : \mathcal{E}_X(\mathcal{F}^{\mathbb{D}}_{\Sigma}) \to \mathbf{E}_X(\mathbf{F}^{\mathbf{D}}_{\Sigma})$ and $(.)^{\bullet} : \mathbf{E}_X(\mathbf{F}^{\mathbf{D}}_{\Sigma}) \to \mathcal{E}_X(\mathcal{F}^{\mathbb{D}}_{\Sigma})$ are well-defined. Since each ideal/eager extension is a complete extension, by Proposition 4, $(.)_{\bullet} : \mathcal{E}_X(\mathcal{F}^{\mathbb{D}}_{\Sigma}) \to \mathbf{E}_X(\mathbf{F}^{\mathbf{D}}_{\Sigma})$ and $(.)^{\bullet} : \mathbf{E}_X(\mathbf{F}^{\mathbf{D}}_{\Sigma}) \to \mathcal{E}_X(\mathcal{F}^{\mathbb{D}}_{\Sigma})$ are injections. By Proposition 6, $(\mathbf{S}^{\bullet})_{\bullet} = \mathbf{S}$ and $(\mathcal{S}_{\bullet})^{\bullet} = \mathcal{S}$. Therefore, $(.)_{\bullet} : \mathcal{E}_X(\mathcal{F}^{\mathbb{D}}_{\Sigma}) \to \mathbf{E}_X(\mathbf{F}^{\mathbf{D}}_{\Sigma})$ and $(.)^{\bullet} : \mathbf{E}_X(\mathbf{F}^{\mathbf{D}}_{\Sigma}) \to \mathcal{E}_X(\mathcal{F}^{\mathbb{D}}_{\Sigma})$ are surjections. $\square$

6.5. Bisimulation implies extensional equivalence

The following is the main theorem in this paper. It shows that bisimulation implies extensional equivalence. X-extensional equivalence (Definition 12) requires that $(.)_{\bullet}$ and $(.)^{\bullet}$ are bijections when restricted to X-extensions. We have proved the related results in the previous subsections. Therefore, the main theorem follows directly.

Theorem 1. *Let $(\mathcal{L}, \vdash)$ be a logic and Σ a knowledge base. Let $\mathcal{F}^{\mathbb{D}}_{\Sigma} = (Ar_b(\Sigma), \mathcal{R}_{\mathbb{D}})$ be a BAF and $\mathbf{F}^{\mathbf{D}}_{\Sigma} = (Ar_p(\Sigma), \mathbf{R}_{\mathbf{D}})$ a PAF. If $\mathcal{F}^{\mathbb{D}}_{\Sigma}$ and $\mathbf{F}^{\mathbf{D}}_{\Sigma}$ are bisimilar, then $\mathcal{F}^{\mathbb{D}}_{\Sigma}$ and $\mathbf{F}^{\mathbf{D}}_{\Sigma}$ are X-extensionally equivalent, where $X \in \{co, st, gr, pr, ss, id, ea\}$.*

Proof. It follows from Propositions 4, 6 and 8. $\square$

For a BAF and a PAF, being extensionally equivalent means that they have the same capacity to evaluate arguments with respect to extensional semantics. Bisimulation provides a sufficient condition for extensional equivalence.

Example 7. The PAF $\mathbf{F}^{\mathbf{D}_{DD}}_{\Sigma}$ in Example 2 and the BAF $\mathcal{F}^{\mathbb{D}_2}_{\Sigma}$ in Example 3 are bisimilar (Example 6). They are shown to be complete-extensionally equivalent in Example 5. By the above theorem, they are X-extensionally equivalent for $X \in \{co, st, gr, pr, ss, id, ea\}$.

Some frames are proved to be bisimilar in Proposition 2. It is a corollary that they are extensionally equivalent.

Corollary 1. *Let $(\mathcal{L}, \vdash)$ be a logic and Σ a knowledge base. For $X \in \{co, st, gr, pr, ss, id, ea\}$,*

(1) $\mathcal{F}^{\mathbb{D}_1}_{\Sigma}$ and $\mathbf{F}^{\mathbf{D}_D}_{\Sigma}$ are X-extensionally equivalent.
(2) $\mathcal{F}^{\mathbb{D}_2}_{\Sigma}$ and $\mathbf{F}^{\mathbf{D}_{DD}}_{\Sigma}$ are X-extensionally equivalent.
(3) $\mathcal{F}^{\mathbb{D}_2}_{\Sigma}$ and $\mathbf{F}^{\mathbf{D}_{DU}}_{\Sigma}$ are X-extensionally equivalent.
(4) $\mathcal{F}^{\mathbb{D}_3}_{\Sigma}$ and $\mathbf{F}^{\mathbf{D}_U}_{\Sigma}$ are X-extensionally equivalent.

Proof. It follows directly from Theorem 1 and Proposition 2. $\square$

Note that $\mathbf{F}^{\mathbf{D}_{DD}}_{\Sigma}$ and $\mathbf{F}^{\mathbf{D}_{DU}}_{\Sigma}$ are extensionally equivalent to the same BAF $\mathcal{F}^{\mathbb{D}_2}_{\Sigma}$ in the above corollary. The following corollary shows that PAFs $\mathbf{F}^{\mathbf{D}_{DD}}_{\Sigma}$ and $\mathbf{F}^{\mathbf{D}_{DU}}_{\Sigma}$ have identical X-extensions, though they have distinct attack relations. The relation between attack functions $\mathbf{D}_{DD}$ and $\mathbf{D}_{DU}$ is shown in Fig. 1. The following result is an example of how base argumentation helps to gain a deeper understanding of premise-conclusion argumentation.

Corollary 2. *Let* $(\mathcal{L},\vdash)$ *be a logic and* Σ *a knowledge base. For* $X \in \{co, st, gr, pr, ss, id, ea\}$*,*

$$\mathbf{E}_X(\mathbf{F}_\Sigma^{\mathbf{D}_{DD}}) = \mathbf{E}_X(\mathbf{F}_\Sigma^{\mathbf{D}_{DU}}).$$

Proof. Assume that $\mathbf{S} \in \mathbf{E}_X(\mathbf{F}_\Sigma^{\mathbf{D}_{DD}})$. By item (2) in Corollary 1, $\mathbf{S}^\bullet \in \mathcal{E}_X(\mathcal{F}_\Sigma^{\mathbb{D}_2})$. By item (3) in Corollary 1, $(\mathbf{S}^\bullet)_\bullet \in \mathbf{E}_X(\mathbf{F}_\Sigma^{\mathbf{D}_{DU}})$. By Lemma 6, $\mathbf{S} = (\mathbf{S}^\bullet)_\bullet$. Therefore, $\mathbf{S} \in \mathbf{E}_X(\mathbf{F}_\Sigma^{\mathbf{D}_{DU}})$.

By a similar proof, $\mathbf{E}_X(\mathbf{F}_\Sigma^{\mathbf{D}_{DU}}) \subseteq \mathbf{E}_X(\mathbf{F}_\Sigma^{\mathbf{D}_{DD}})$. □

To further demonstrate how base argumentation, bisimulation and extensional equivalence can contribute to the study of premise-conclusion argumentation, we conduct in-depth research on the extensional properties of $\mathcal{F}_\Sigma^{\mathbb{D}_2}$ in the next section and transfer these properties to $\mathbf{F}_\Sigma^{\mathbf{D}_{DD}}$ and $\mathbf{F}_\Sigma^{\mathbf{D}_{DU}}$ via bisimulation and extensional equivalence in Section 8.

7. Extensional properties of $\mathcal{F}_\Sigma^{\mathbb{D}_2}$

The main result in this section is that there are essentially three kinds of extensions in $\mathcal{F}_\Sigma^{\mathbb{D}_2}$: complete, preferred and grounded extensions (Propositions 13 and 15). These properties are exported to $\mathbf{F}_\Sigma^{\mathbf{D}_{DD}}$ and $\mathbf{F}_\Sigma^{\mathbf{D}_{DU}}$ via bisimulation and extensional equivalence in the next section. For an example of $\mathcal{F}_\Sigma^{\mathbb{D}_2}$, see Example 4.

This section assumes as given an abstract logic $(\mathcal{L},\vdash)$ and a knowledge base Σ. Since our main concern is $\mathcal{F}_\Sigma^{\mathbb{D}_2}$, $\mathbb{D}_2$ is omitted when no confusion arises, e.g., we write "Γ attacks Δ" instead of "Γ $\mathbb{D}_2$-attacks Δ".

7.1. The form of complete extensions

By the definition of base arguments (Definition 8), not every subset of the knowledge base is a base argument. Therefore we introduce a notation for the set of base arguments which are subsets of a given set.

Definition 14. Let $\Phi \subseteq \Sigma$. Denote by $\rho(\Phi)$ the set of base arguments which are subsets of Φ, i.e., $\rho(\Phi)=\{\Gamma \subseteq \Phi \mid \Gamma \text{ is a base argument}\}$.

Lemma 2 shows that any subset of a base argument is a base argument. It follows that for any base argument Γ, each element in $\mathcal{P}(\Gamma)$ is a base argument, where $\mathcal{P}(\Gamma)$ is the power set of Γ.

The following lemma shows that each complete extension in $\mathcal{F}_\Sigma^{\mathbb{D}_2}$ contains the empty set, and is closed under union and intersection.

Lemma 8. *Let* $\mathcal{S}$ *be a complete extension in* $\mathcal{F}_\Sigma^{\mathbb{D}_2}$*.*

1. $\emptyset \in \mathcal{S}$.
2. *For any* $\mathcal{S}' \subseteq \mathcal{S}$*, if* $\bigcup_{\Gamma \in \mathcal{S}'} \Gamma$ *is a base argument, then* $\bigcup_{\Gamma \in \mathcal{S}'} \Gamma \in \mathcal{S}$.
3. *For any* $\mathcal{S}' \subseteq \mathcal{S}$*, if* $\bigcap_{\Gamma \in \mathcal{S}'} \Gamma$ *is a base argument, then* $\bigcap_{\Gamma \in \mathcal{S}'} \Gamma \in \mathcal{S}$.

Proof. (1) By definition, $\emptyset$ is not attacked by any base argument. Then $\emptyset$ is defended by any extension. Since $\mathcal{S}$ is a complete extension, $\emptyset \in \mathcal{S}$.

(2) It suffices to show that $\mathcal{S}$ defends $\bigcup_{\Gamma \in \mathcal{S}'} \Gamma$. Assume that Ω attacks $\bigcup_{\Gamma \in \mathcal{S}'} \Gamma$. Then there exists $\varphi \in \bigcup_{\Gamma \in \mathcal{S}'} \Gamma$ such that $\Omega \vdash \neg\varphi$. Then there exists $\Gamma' \in \mathcal{S}'$ such that $\varphi \in \Gamma'$. Since $\mathcal{S}' \subseteq \mathcal{S}$, $\Gamma' \in \mathcal{S}$. Since $\Omega \vdash \neg\varphi$ and $\varphi \in \Gamma'$, Ω attacks Γ'. Since $\mathcal{S}$ is complete, there exists $\Omega' \in \mathcal{S}$ that attacks Ω. Therefore, $\mathcal{S}$ defends $\bigcup_{\Gamma \in \mathcal{S}'} \Gamma$.

(3) It follows from the sub-argument closure property (Lemma 4). □

The following proposition shows that each complete extension in $\mathcal{F}_\Sigma^{\mathbb{D}_2}$ is of a certain form.

Proposition 9. *If* $\mathcal{S}$ *is a complete extension in* $\mathcal{F}_\Sigma^{\mathbb{D}_2}$*, then* $\mathcal{S} = \rho(\bigcup_{\Gamma \in \mathcal{S}} \Gamma)$.

Proof. Assume that $\Delta \in \mathcal{S}$. Then $\Delta \subseteq \bigcup_{\Gamma \in \mathcal{S}} \Gamma$. Since Δ is a base argument, $\Delta \in \rho(\bigcup_{\Gamma \in \mathcal{S}} \Gamma)$.

Assume that $\Delta \in \rho(\bigcup_{\Gamma \in \mathcal{S}} \Gamma)$. Then Δ is a base argument and $\Delta \subseteq \bigcup_{\Gamma \in \mathcal{S}} \Gamma$. It follows that $\Delta = \bigcup_{\Gamma \in \mathcal{S}}(\Delta \cap \Gamma)$. For any $\Gamma \in \mathcal{S}$, by Lemma 2 and the sub-argument closure property (Lemma 4), $\Delta \cap \Gamma$ is a base argument $\Delta \cap \Gamma \in \mathcal{S}$. It follows that $\{\Delta \cap \Gamma \mid \Gamma \in \mathcal{S}\} \subseteq \mathcal{S}$. Since $\mathcal{S}$ is a complete extension, by Lemma 8, $\bigcup_{\Gamma \in \mathcal{S}}(\Delta \cap \Gamma) \in \mathcal{S}$, i.e., $\Delta \in \mathcal{S}$. □

7.2. Two typical complete extensions

This subsection shows that $\rho(\Gamma)$ and $\rho(\bigcap \mathbf{MC}(\Sigma))$ are complete extensions in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$, where Γ is a maximal consistent subset of the knowledge base Σ and $\bigcap \mathbf{MC}(\Sigma)$ denotes the intersection of maximal consistent subsets of Σ.

Lemma 9. *Let Γ be a maximal consistent subset of Σ. If $\varphi \in \Sigma$ and $\varphi \notin \Gamma$, then $\Gamma \vdash \neg\varphi$.*

Proof. Since $\varphi \in \Sigma$ and $\varphi \notin \Gamma$, $\Gamma \cup \{\varphi\}$ is inconsistent. By definition, $\Gamma \cup \{\varphi\} \vdash \bot$. By the property of negation, $\Gamma \vdash \neg\varphi$. □

Lemma 10. *Let Γ be a maximal consistent subset of Σ such that $\Gamma \vdash \varphi$. Then there exists a base argument $\Delta \subseteq \Gamma$ such that $\Delta \vdash \varphi$.*

Proof. Since $\Gamma \vdash \varphi$ and $\vdash$ is finitary, there exists a finite subset Γ' of Γ such that $\Gamma \vdash \varphi$. Let $\Gamma' = \{\varphi_1, \dots, \varphi_n\}$. Add the formulas in Γ' one by one to Δ until $\Delta \vdash \varphi$. It follows that there does not exist $\Delta' \subset \Delta$ such that $\Delta' \vdash \varphi$. Therefore, there does not exist $\Delta' \subset \Delta$ such that $Cn_{\vdash}(\Delta') = Cn_{\vdash}(\Delta)$. It follows that Δ is a base argument. □

The following proposition shows that $\rho(\Gamma)$ is a complete extension in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$, where Γ is a maximal consistent subset of the knowledge base.

Proposition 10. *Let Γ be a maximal consistent subset of Σ. Then $\mathcal{S} = \rho(\Gamma)$ is a complete extension in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$.*

Proof. First we show that $\mathcal{S}$ is conflict-free. Assume that $\Delta, \Delta' \in \mathcal{S}$ and that Δ attacks Δ'. Then there exists $\varphi \in \Delta'$ such that $\Delta \vdash \neg\varphi$. Since $\Delta, \Delta' \in \mathcal{S}$, $\Delta, \Delta' \subseteq \Gamma$. By the monotonicity of $\vdash$, $\Gamma \vdash \neg\varphi$. Since $\Delta' \subseteq \Gamma$, $\varphi \in \Gamma$, and hence $\Gamma \vdash \varphi$. Therefore, Γ is inconsistent, contradiction. It follows that $\mathcal{S}$ is conflict-free.

Now we show that $\mathcal{S}$ defends each of its elements. Assume that $\Delta \in \mathcal{S}$ and that Ω attacks Δ. Then there exists $\varphi \in \Delta$ such that $\Omega \vdash \neg\varphi$. Since $\Delta \subseteq \Gamma$, $\varphi \in \Gamma$. Since Γ is consistent and $\Omega \vdash \neg\varphi$, by the monotonicity of $\vdash$, $\Omega \nsubseteq \Gamma$. Let ψ be a formula such that $\psi \in \Omega$ and $\psi \notin \Gamma$. Since Ω is a base argument, $\Omega \subseteq \Sigma$. Then $\psi \in \Omega$. By Lemma 9, $\Gamma \vdash \neg\psi$. Therefore, Γ attacks Ω.

Next we show that $\mathcal{S}$ contains each argument it defends. Assume that $\mathcal{S}$ defends a base argument Δ. It suffices to show that $\Delta \subseteq \Gamma$. We prove by contradiction and assume that $\varphi \in \Delta$ and $\varphi \notin \Gamma$. Since Δ is a base argument, $\Delta \subseteq \Sigma$. Then $\varphi \in \Sigma$. By Lemma 9, $\Gamma \vdash \neg\varphi$. By Lemma 10, there exists a base argument $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash \neg\varphi$. It follows that $\Gamma' \in \mathcal{S}$. Therefore, Γ' attacks Δ. Since $\mathcal{S}$ defends Δ, there exists $\Gamma'' \in \mathcal{S}$ that attacks Γ'. Since $\Gamma' \in \mathcal{S}$, $\mathcal{S}$ is not conflict-free, contradiction. □

Now we start to show that $\rho(\bigcap \mathbf{MC}(\Sigma))$ is a complete extension in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$, where $\mathbf{MC}(\Sigma)$ denotes the set of maximal consistent subsets of Σ and $\bigcap \mathbf{MC}(\Sigma)$ denotes the intersection of sets in $\mathbf{MC}(\Sigma)$.

First we define an operation to obtain a maximal consistent subset from a consistent subset of the knowledge base.

Definition 15. Let Γ be a consistent subset of Σ. Denote by $Max(\Gamma)$ the maximal consistent subset of Σ generated as follows: let $\varphi_1, \varphi_2, \dots$ be a enumeration of formulas in $\Sigma \setminus \Gamma$,

$$\Gamma_0 = \Gamma$$

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\varphi_n\}, & \text{if it is consistent} \\ \Gamma_n, & \text{otherwise} \end{cases}$$

$$Max(\Gamma) = \bigcup_{n \geq 0} \Gamma_n.$$

It can be proved by induction on n that Definition 15 is well-defined.

Corollary 3. *Let Γ be a base argument. Then $\mathcal{S} = \rho(Max(\Gamma))$ is a complete extension in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$.*

Proof. It follows directly from Proposition 10. □

The following are some properties about $\bigcap \mathbf{MC}(\Sigma)$.

Lemma 11. *Let $(\mathcal{L}, \vdash)$ be a logic and Σ a knowledge base.*

1. *$\bigcap \mathbf{MC}(\Sigma)$ is consistent.*
2. *For any $\Gamma \in \rho(\bigcap \mathbf{MC}(\Sigma))$, no base argument $\mathbb{D}_2$-attacks Γ.*
3. *For any $\Gamma \in \rho(\bigcap \mathbf{MC}(\Sigma))$, Γ does not $\mathbb{D}_2$-attack any base argument.*
4. *For any base argument Γ, if $\Gamma \notin \rho(\bigcap \mathbf{MC}(\Sigma))$, then Γ is $\mathbb{D}_2$-attacked by a base argument.*

Proof. (1) It follows directly from the definitions.

(2) We prove by contradiction and assume that a base argument Δ attacks Γ. Then there exists $\varphi \in \Gamma$ such that $\Delta \vdash \neg\varphi$. Since $\Gamma \subseteq \bigcap \mathbf{MC}(\Sigma)$, $\varphi \in \bigcap \mathbf{MC}(\Sigma)$. Therefore, $\varphi \in Max(\Delta)$. Since $\Delta \vdash \neg\varphi$ and $\Delta \subseteq Max(\Delta)$, by the monotonicity of $\vdash$, $Max(\Delta) \vdash \neg\varphi$. Since $\varphi \in Max(\Delta)$, $Max(\Delta)$ is inconsistent, contradiction. Therefore, no base argument attacks Γ.

(3) We prove by contradiction and assume that Γ attacks a base argument Δ. Then there exists $\varphi \in \Delta$ such that $\Gamma \vdash \neg\varphi$. Since $\Delta \subseteq Max(\Delta)$, $\varphi \in Max(\Delta)$. Since $\Gamma \subseteq \bigcap \mathbf{MC}(\Sigma) \subseteq Max(\Delta)$, by the monotonicity of $\vdash$, $Max(\Delta) \vdash \neg\varphi$. It follows that $Max(\Delta)$ is inconsistent, contradiction. Therefore, Γ does not attack any base argument.

(4) Since $\Gamma \notin \rho(\bigcap \mathbf{MC}(\Sigma))$, $\Gamma \nsubseteq \bigcap \mathbf{MC}(\Sigma)$. Then there exists $\varphi \in \Gamma$ such that $\varphi \notin \bigcap \mathbf{MC}(\Sigma)$. Then there exists $\Delta \in \mathbf{MC}(\Sigma)$ such that $\varphi \notin \Delta$. By Lemma 9, $\Delta \vdash \neg\varphi$. By Lemma 10, there exists a base argument $\Delta' \subseteq \Delta$ that attacks Γ. □

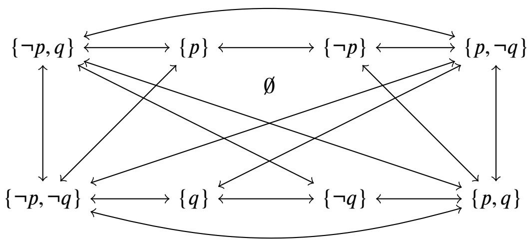
The following proposition shows that $\rho(\bigcap \mathbf{MC}(\Sigma))$ is a complete extension in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$.

Proposition 11. *$\mathcal{S} = \rho(\bigcap \mathbf{MC}(\Sigma))$ is a complete extension in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$.*

Proof. By items (1) and (2) in Lemma 11, $\mathcal{S}$ is conflict-free and defends each of its elements. Assume that $\mathcal{S}$ defends Γ. To show that $\Gamma \in \mathcal{S}$, it suffices to show that $\Gamma \subseteq \bigcap \mathbf{MC}(\Sigma)$. Assume that $\Gamma \nsubseteq \bigcap \mathbf{MC}(\Sigma)$, by item (4) in Lemma 11, Γ is attacked by some base argument Δ. Since $\mathcal{S}$ defends Γ, $\mathcal{S}$ attacks Γ, contradicting item (3) in Lemma 11 (3). Therefore, $\mathcal{S}$ is a complete extension. □

Remark 1. It is not the case that for any consistent subset Δ of Σ, $\rho(\Delta)$ is a complete extension. In Example 3, $\{p\}$ is a consistent subset of the knowledge base, but $\{\emptyset, \{p\}\}$ is not a complete extension.

Remark 2. It is not the case that each complete extension is of the form $\rho(\Gamma)$ or $\rho(\bigcap \mathbf{MC}(\Sigma))$, where Γ is a maximal consistent subset of Σ. Consider the following example: let $(\mathcal{L}, \vdash)$ be propositional logic and $\Sigma = \{p, \neg p, q, \neg q\}$. The generated BAF is as follows:



In this example, $\mathbf{MC}(\Sigma) = \{\{\neg p, q\}, \{p, \neg q\}, \{\neg p, \neg q\}, \{p, q\}\}$, $\bigcap \mathbf{MC}(\Sigma) = \emptyset$. Note that $\{\emptyset, \{p\}\}$ is a complete extension but it is not of the form $\rho(\bigcap \mathbf{MC}(\Sigma))$ or $\rho(\Gamma)$, where $\Gamma \in \mathbf{MC}(\Sigma)$.

7.3. Preferred, stable and semi-stable extensions are identical

This subsection shows that preferred, stable and semi-stable extensions are identical in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$.

The following proposition shows that preferred extensions in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$ have a closed relation with the maximal consistent subsets of the knowledge base.

Proposition 12. *For any $\mathcal{S} \subseteq Ar_b(\Sigma)$, $\mathcal{S}$ is a preferred extension in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$ iff there exists $\Gamma \in \mathbf{MC}(\Sigma)$ such that $\mathcal{S} = \rho(\Gamma)$.*

Proof. For the left-to-right direction, assume that $\mathcal{S}$ is a preferred extension in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$. Since each preferred extension is a complete extension, by Proposition 9, $\mathcal{S} = \rho(\bigcup_{\Gamma \in \mathcal{S}} \Gamma)$. By Proposition 10, $\mathcal{S}' = \rho(Max(\bigcup_{\Gamma \in \mathcal{S}} \Gamma))$ is a complete extension. Since $\bigcup_{\Gamma \in \mathcal{S}} \Gamma \subseteq Max(\bigcup_{\Gamma \in \mathcal{S}} \Gamma)$, $\mathcal{S} \subseteq \mathcal{S}'$. Since $\mathcal{S}$ is a preferred extension, $\mathcal{S} = \mathcal{S}' = \rho(Max(\bigcup_{\Gamma \in \mathcal{S}} \Gamma))$, $Max(\bigcup_{\Gamma \in \mathcal{S}} \Gamma)$ is the required maximal consistent subset of Σ.

Now we consider the right-to-left direction. By Proposition 10, $\mathcal{S} = \rho(\Gamma)$ is a complete extension in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$. We need to show that $\mathcal{S}$ is a maximal complete extension. Let $\mathcal{S}'$ be a complete extension such that $\mathcal{S} \subseteq \mathcal{S}'$. By Proposition 9, $\mathcal{S}' = \rho(\bigcup_{\Delta \in \mathcal{S}'} \Delta)$.

Now we show that $\mathcal{S}' \subseteq \mathcal{S}$, i.e., $\rho(\bigcup_{\Delta \in \mathcal{S}'} \Delta) \subseteq \rho(\Gamma)$. Assume that $\rho(\bigcup_{\Delta \in \mathcal{S}'} \Delta) \nsubseteq \rho(\Gamma)$. Then there exists a base argument $\Omega \subseteq \bigcup_{\Delta \in \mathcal{S}'} \Delta$ such that $\Omega \nsubseteq \Gamma$. Then there exists $\varphi \in \Omega$ such that $\varphi \notin \Gamma$. Since Ω is a base argument, $\Omega \subseteq \Sigma$. Then $\varphi \in \Sigma$. By Lemma 9, $\Gamma \vdash \neg\varphi$. By Lemma 10, there exists a base argument $\Gamma' \subseteq \Gamma$ such that $\Gamma \vdash \neg\varphi$. It follows that Γ' attacks Ω. Since $\mathcal{S} = \rho(\Gamma)$ and $\Gamma' \subseteq \Gamma$, $\Gamma' \in \mathcal{S}$. Since $\mathcal{S} \subseteq \mathcal{S}'$, $\Gamma' \in \mathcal{S}'$. Since $\Omega \in \mathcal{S}'$, $\mathcal{S}'$ is not conflict-free, contradicting the fact that $\mathcal{S}'$ is complete. Therefore, $\mathcal{S}' \subseteq \mathcal{S}$. It follows that $\mathcal{S}$ is a maximal complete extension. □

Recall that $\mathcal{S}^+$ denotes the set of base arguments attacked by $\mathcal{S}$, i.e., $\mathcal{S}^+ := \{\Gamma \in Ar_b(\Sigma) \mid \mathcal{S} \text{ attacks } \Gamma\}$. The following lemma shows that $\mathcal{S} \cup \mathcal{S}^+ = Ar_b(\Sigma)$ for any preferred extension $\mathcal{S}$ in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$.

Lemma 12. *If $\mathcal{S}$ is a preferred extension in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$, then $\mathcal{S} \cup \mathcal{S}^+ = Ar_b(\Sigma)$.*

Proof. Since $\mathcal{S} \subseteq Ar_b(\Sigma)$ and $\mathcal{S}^+ \subseteq Ar_b(\Sigma)$, $\mathcal{S} \cup \mathcal{S}^+ \subseteq Ar_b(\Sigma)$.

Now we show that $Ar_b(\Sigma) \subseteq \mathcal{S} \cup \mathcal{S}^+$. Assume that $\Omega \in Ar_b(\Sigma)$ and $\Omega \notin \mathcal{S}$. Since $\mathcal{S}$ is a preferred extension in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$, by Proposition 12, there exists $\Gamma \in \mathbf{MC}(\Sigma)$ such that $\mathcal{S} = \rho(\Gamma)$. Since $\Omega \notin \mathcal{S}$, $\Omega \nsubseteq \Gamma$. It follows that there exists $\varphi \in \Omega$ such that $\varphi \notin \Gamma$. Since $\Omega \in Ar_b(\Sigma)$, $\Omega \subseteq \Sigma$. Then $\varphi \in \Sigma$. By Lemma 9, $\Gamma \vdash \neg\varphi$. By Lemma 10, there exists a base argument $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash \neg\varphi$. Therefore, Γ' attacks Ω. Since $\mathcal{S} = \rho(\Gamma)$, $\Gamma' \in \mathcal{S}$. It follows that $\Omega \in \mathcal{S}^+$. □

The following proposition shows that preferred, stable and semi-stable extensions are identical in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$.

Proposition 13. *Let $\mathcal{S} \subseteq Ar_b(\Sigma)$. The following conditions are equivalent:*

1. *$\mathcal{S}$ is a preferred extension in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$.*
2. *$\mathcal{S}$ is a stable extension in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$.*
3. *$\mathcal{S}$ is a semi-stable extension in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$.*

Proof. (1) ⇒ (2): Assume that $\mathcal{S}$ is a preferred extension in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$. By Proposition 12, there exists $\Gamma \in \mathbf{MC}(\Sigma)$ such that $\mathcal{S} = \rho(\Gamma)$. To show that $\mathcal{S}$ is stable, we need to show that $\mathcal{S}$ attacks each argument that is not in $\mathcal{S}$. By Lemma 12, $\mathcal{S} \cup \mathcal{S}^+ = Ar_b(\Sigma)$. It follows that $\mathcal{S}$ attacks each argument that is not in $\mathcal{S}$.

(2) ⇒ (3): It follows from the fact that each stable extension is a semi-stable extension.

(3) ⇒ (1): It follows from the fact that each semi-stable extension is a preferred extension. □

7.4. Grounded, ideal and eager extensions are identical

This subsection shows that grounded, ideal and eager extensions are identical in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$.

The following proposition shows that the grounded extension in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$ is closely related to the intersection of all maximal consistent subsets of the knowledge base.

Proposition 14. *$\mathcal{S} = \rho(\bigcap \mathbf{MC}(\Sigma))$ is the grounded extension in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$.*

Proof. By Proposition 11, $\mathcal{S}$ is a complete extension. We need to show that for any complete extension $\mathcal{S}'$, $\mathcal{S} \subseteq \mathcal{S}'$. Let $\Gamma \in \mathcal{S}$. By Lemma 11, Γ is not attacked. Therefore, $\mathcal{S}'$ defends Γ. Since $\mathcal{S}'$ is a complete extension, $\Gamma \in \mathcal{S}'$. □

The following lemma shows that ρ distributes over intersection.

Lemma 13. *Let Φ be a set of consistent subsets of Σ. Then*

$$\bigcap_{\Gamma \in \Phi} \rho(\Gamma) = \rho(\bigcap_{\Gamma \in \Phi} \Gamma).$$

Proof. Assume that $\Delta \in \bigcap_{\Gamma \in \Phi} \rho(\Gamma)$. Then Δ is a base argument by Lemma 2 and for any $\Gamma \in \Phi$, $\Delta \subseteq \Gamma$. Then $\Delta \subseteq \bigcap_{\Gamma \in \Phi} \Gamma$. It follows that $\Delta \in \rho(\bigcap_{\Gamma \in \Phi} \Gamma)$.

Assume that $\Delta \in \rho(\bigcap_{\Gamma \in \Phi} \Gamma)$. Then Δ is a base argument and $\Delta \subseteq \bigcap_{\Gamma \in \Phi} \Gamma$. Then for any $\Gamma \in \Phi$, $\Delta \in \rho(\Gamma)$. Therefore, $\Delta \in \bigcap_{\Gamma \in \Phi} \rho(\Gamma)$. □

The following proposition shows that grounded, ideal and eager extensions are identical in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$.

Proposition 15. *Let $(\mathcal{L}, \vdash)$ be a logic and Σ a knowledge base. Let $\mathcal{S} \subseteq Ar_b(\Sigma)$. Then the following conditions are equivalent:*

1. *$\mathcal{S}$ is the grounded extension in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$.*
2. *$\mathcal{S}$ is an ideal extension in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$.*
3. *$\mathcal{S}$ is an eager extension in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$.*

Proof. (1) ⇒ (2): Assume that $\mathcal{S}$ is the grounded extension. By Proposition 14, $\mathcal{S} = \rho(\bigcap \mathbf{MC}(\Sigma))$. To show that $\mathcal{S}$ is an ideal extension, we need to show that $\mathcal{S}$ is a maximal complete extension contained in each preferred extension.

- Since each grounded extension is a complete extension, $\mathcal{S}$ is a complete extension.

– By Proposition 12, for any preferred extension $\mathcal{S}'$, there exists $\Gamma \in \mathbf{MC}(\Sigma)$ such that $\mathcal{S}' = \rho(\Gamma)$. Since for any $\Gamma \in \mathbf{MC}(\Sigma)$, $\rho(\bigcap \mathbf{MC}(\Sigma)) \subseteq \rho(\Gamma)$, $\mathcal{S} \subseteq \mathcal{S}'$ for any preferred extension $\mathcal{S}'$.
– Assume that $\mathcal{S}'$ is a complete extension that is contained in each preferred extension such that $\mathcal{S} \subseteq \mathcal{S}'$. By Proposition 12, for any $\Gamma \in \mathbf{MC}(\Sigma)$, $\mathcal{S}' \subseteq \rho(\Gamma)$. Therefore, $\mathcal{S}' \subseteq \bigcap_{\Gamma \in \mathbf{MC}(\Sigma)} \rho(\Gamma)$. By Lemma 13, $\bigcap_{\Gamma \in \mathbf{MC}(\Sigma)} \rho(\Gamma) = \rho(\bigcap \mathbf{MC}(\Sigma))$. It follows that $\mathcal{S}' \subseteq \mathcal{S}$. Since $\mathcal{S} \subseteq \mathcal{S}'$, $\mathcal{S} = \mathcal{S}'$.

Therefore, $\mathcal{S}$ is an ideal extension.

(2) ⇒ (1): Assume that $\mathcal{S}$ is an ideal extension. Then $\mathcal{S}$ is a maximal complete extension that is contained in each preferred extension. By Proposition 14, we need to show that $\mathcal{S} = \rho(\bigcap \mathbf{MC}(\Sigma))$. By Proposition 12, a set $\mathcal{S}'$ of base arguments is a preferred extension iff there exists $\Gamma \in \mathbf{MC}(\Sigma)$ such that $\mathcal{S}' = \rho(\Gamma)$. Therefore, $\{\rho(\Gamma) \mid \Gamma \in \mathbf{MC}(\Sigma)\}$ is the set of all preferred extensions. Since $\mathcal{S}$ is contained in each preferred extension, $\mathcal{S} \subseteq \bigcap\{\rho(\Gamma) \mid \Gamma \in \mathbf{MC}(\Sigma)\}$. By Lemma 13, $\bigcap\{\rho(\Gamma) \mid \Gamma \in \mathbf{MC}(\Sigma)\} = \rho(\bigcap \mathbf{MC}(\Sigma))$. Therefore, $\mathcal{S} \subseteq \rho(\bigcap \mathbf{MC}(\Sigma))$. By Proposition 14, $\rho(\bigcap \mathbf{MC}(\Sigma))$ is the grounded extension. Therefore, $\rho(\bigcap \mathbf{MC}(\Sigma)) \subseteq \mathcal{S}$. It follows that $\mathcal{S} = \rho(\bigcap \mathbf{MC}(\Sigma))$.

(1) ⇒ (3): We have the following biconditionals:

		$\mathcal{S}$ is contained in each semi-stable extension.
(Proposition 13)	iff	$\mathcal{S}$ is contained in each preferred extension
(Proposition 12)	iff	For any $\Gamma \in \mathbf{MC}(\Sigma)$, $\mathcal{S} \subseteq \rho(\Gamma)$
	iff	$\mathcal{S} \subseteq \bigcap_{\Gamma \in \mathbf{MC}(\Sigma)} \rho(\Gamma)$
(Lemma 13)	iff	$\mathcal{S} \subseteq \rho(\bigcap \mathbf{MC}(\Sigma))$

Therefore, to show that $\mathcal{S}$ is an eager extension, it suffices to show that $\mathcal{S}$ is a maximal complete extension such that $\mathcal{S} \subseteq \rho(\bigcap \mathbf{MC}(\Sigma))$. Since $\mathcal{S}$ is the grounded extension, by Proposition 14, $\mathcal{S} = \rho(\bigcap \mathbf{MC}(\Sigma))$, which completes the proof.

(3) ⇒ (1): Assume that $\mathcal{S}$ is an eager extension. By the biconditionals in the proof of (1) ⇒ (3), $\mathcal{S}$ is a maximal complete extension such that $\mathcal{S} \subseteq \rho(\bigcap \mathbf{MC}(\Sigma))$. By Proposition 11, $\rho(\bigcap \mathbf{MC}(\Sigma))$ is a complete extension. Since $\mathcal{S} \subseteq \rho(\bigcap \mathbf{MC}(\Sigma))$, by the maximality of $\mathcal{S}$, $\mathcal{S} = \rho(\bigcap \mathbf{MC}(\Sigma))$. By Proposition 14, $\mathcal{S}$ is the grounded extension. □

8. Extensional properties of $\mathbf{F}_{\Sigma}^{\mathbf{D}_{DD}}$ and $\mathbf{F}_{\Sigma}^{\mathbf{D}_{DU}}$

This section exports the extensional properties of $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$ investigated in the last section to $\mathbf{F}_{\Sigma}^{\mathbf{D}_{DD}}$ and $\mathbf{F}_{\Sigma}^{\mathbf{D}_{DU}}$ via bisimulation and extensional equivalence. Since the extensions in $\mathbf{F}_{\Sigma}^{\mathbf{D}_{DD}}$ and $\mathbf{F}_{\Sigma}^{\mathbf{D}_{DU}}$ are identical (Corollary 2), they are essentially one PAF from the perspective of extensions.

For a knowledge base Σ, denote by $\mu(\Gamma)$ the set of premise-conclusion arguments whose premise set is a subset of Γ, i.e.,

$$\mu(\Gamma) := \{(\Gamma', \varphi) \mid (\Gamma', \varphi) \in Ar_p(\Sigma) \text{ and } \Gamma' \subseteq \Gamma\}.$$

The following lemma relates $\mu(\Gamma)$ to $\rho(\Gamma)$ by $(.)_{\bullet}$. It shows that the set of premise-conclusion arguments whose premise set is a subset of Γ is identical to the result of applying $(.)_{\bullet}$ to the set of base arguments which are subsets of Γ.

Lemma 14. *Let $(\mathcal{L}, \vdash)$ be an abstract logic and Σ a knowledge base. Let Γ be a consistent subset of Σ. Then $(\rho(\Gamma))_{\bullet} = \mu(\Gamma)$.*

Proof. By definition,

$$(\rho(\Gamma))_{\bullet} = \{(\Gamma', \varphi) \mid (\Gamma', \varphi) \in Ar_p(\Sigma) \text{ and } \exists \Gamma'' \in \rho(\Gamma), \Gamma' \subseteq \Gamma''\}.$$

Assume that $(\Delta, \psi) \in (\rho(\Gamma))_{\bullet}$. Since $\rho(\Gamma) = \{\Gamma'' \mid \Gamma'' \in Ar_b(\Sigma) \text{ and } \Gamma'' \subseteq \Gamma\}$, $\Delta \subseteq \Gamma$. Therefore, $(\Delta, \psi) \in \mu(\Gamma)$.

Assume that $(\Delta, \psi) \in \mu(\Gamma)$. Then (Δ, ψ) is a premise-conclusion argument and $\Delta \subseteq \Gamma$. By Lemma 3, Δ is a base argument. Therefore, $\Delta \in \rho(\Gamma)$. It follows that $(\Delta, \psi) \in (\rho(\Gamma))_{\bullet}$. □

The following proposition shows that preferred extensions and the grounded extension in $\mathbf{F}_{\Sigma}^{\mathbf{D}_{DD}}$ and $\mathbf{F}_{\Sigma}^{\mathbf{D}_{DU}}$ are closely related to the maximal consistent subsets of the knowledge base. It is proved with the relevant results in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$ in the last section via extensional equivalence.

Proposition 16. *Let $(\mathcal{L}, \vdash)$ be an abstract logic and Σ a knowledge base.*

1. *For any $\mathbf{S} \subseteq Ar_p(\Sigma)$, $\mathbf{S}$ is a preferred extension in $\mathbf{F}_{\Sigma}^{\mathbf{D}_{DD}}$ (or $\mathbf{F}_{\Sigma}^{\mathbf{D}_{DU}}$) iff there exists $\Gamma \in \mathbf{MC}(\Sigma)$ such that $\mathbf{S} = \mu(\Gamma)$.*
2. *$\mathbf{S} = \mu(\bigcap \mathbf{MC}(\Sigma))$ is the grounded extension in $\mathbf{F}_{\Sigma}^{\mathbf{D}_{DD}}$ (or $\mathbf{F}_{\Sigma}^{\mathbf{D}_{DU}}$).*

Proof. (1) Since $\mathcal{F}_\Sigma^{\mathbb{D}_2}$ and $\mathbf{F}_\Sigma^{\mathbf{D}_{DD}}$ (or $\mathbf{F}_\Sigma^{\mathbf{D}_{DU}}$) are bisimilar (Proposition 2), the following biconditionals hold:

$$
\begin{aligned}
& & & \mathbf{S} \text{ is a preferred extension in } \mathbf{F}_\Sigma^{\mathbf{D}_{DD}} \text{ (or } \mathbf{F}_\Sigma^{\mathbf{D}_{DU}}) \\
\text{(Corollary 1)} & & \text{iff} \quad & \mathbf{S}^\bullet \text{ is a preferred extension in } \mathcal{F}_\Sigma^{\mathbb{D}_2} \\
\text{(Proposition 12)} & & \text{iff} \quad & \exists \Gamma \in \mathbf{MC}(\Sigma), \mathbf{S}^\bullet = \rho(\Gamma) \\
\text{(Item (2) in Lemma 6)} & & \text{iff} \quad & \exists \Gamma \in \mathbf{MC}(\Sigma), (\mathbf{S}^\bullet)_\bullet = (\rho(\Gamma))_\bullet \\
(*) & & \text{iff} \quad & \exists \Gamma \in \mathbf{MC}(\Sigma), \mathbf{S} = (\rho(\Gamma))_\bullet \\
\text{(Lemma 14)} & & \text{iff} \quad & \exists \Gamma \in \mathbf{MC}(\Sigma), \mathbf{S} = \mu(\Gamma).
\end{aligned}
$$

The top-to-bottom direction of $(*)$ follows from Item (3) in Lemma 6 and the other direction follows from the fact that $(.)_\bullet$ is a bijection.

(2) By Corollary 1, it suffices to show that $(\mu(\bigcap \mathbf{MC}(\Sigma)))^\bullet$ is the grounded extension in $\mathcal{F}_\Sigma^{\mathbb{D}_2}$. By Proposition 14, $\rho(\bigcap \mathbf{MC}(\Sigma))$ is the grounded extension in $\mathcal{F}_\Sigma^{\mathbb{D}_2}$. By Lemma 14, $(\rho(\bigcap \mathbf{MC}(\Sigma)))_\bullet = \mu(\bigcap \mathbf{MC}(\Sigma))$. By Lemma 6,

$$\rho(\bigcap \mathbf{MC}(\Sigma)) = ((\rho(\bigcap \mathbf{MC}(\Sigma)))_\bullet)^\bullet = (\mu(\bigcap \mathbf{MC}(\Sigma)))^\bullet.$$

It follows that $(\mu(\bigcap \mathbf{MC}(\Sigma)))^\bullet$ is the grounded extension in $\mathcal{F}_\Sigma^{\mathbb{D}_2}$. □

The following proposition shows that preferred extensions, stable extensions and semi-stable extensions are identical in $\mathbf{F}_\Sigma^{\mathbf{D}_{DD}}$ and $\mathbf{F}_\Sigma^{\mathbf{D}_{DU}}$. It is proved with Proposition 13 for $\mathcal{F}_\Sigma^{\mathbb{D}_2}$ via extensional equivalence.

Proposition 17. *Let $(\mathcal{L}, \vdash)$ be an abstract logic, Σ a knowledge base. Let $\mathbf{S} \subseteq Ar_p(\Sigma)$. Then the following statements are equivalent:*

1. *$\mathbf{S}$ is a preferred extension in $\mathbf{F}_\Sigma^{\mathbf{D}_{DD}}/\mathbf{F}_\Sigma^{\mathbf{D}_{DU}}$.*
2. *$\mathbf{S}$ is a stable extension in $\mathbf{F}_\Sigma^{\mathbf{D}_{DD}}/\mathbf{F}_\Sigma^{\mathbf{D}_{DU}}$.*
3. *$\mathbf{S}$ is a semi-stable extension in $\mathbf{F}_\Sigma^{\mathbf{D}_{DD}}/\mathbf{F}_\Sigma^{\mathbf{D}_{DU}}$.*

Proof. Assume that $\mathbf{S}$ is a preferred extension in $\mathbf{F}_\Sigma^{\mathbf{D}_{DD}}$. By Corollary 1, $\mathbf{F}_\Sigma^{\mathbf{D}_{DD}}$ and $\mathcal{F}_\Sigma^{\mathbb{D}_2}$ are X-extensionally equivalent, where $X \in \{co, st, gr, pr, ss, id, ea\}$. Therefore, $\mathbf{S}^\bullet$ is a preferred extension in $\mathcal{F}_\Sigma^{\mathbb{D}_2}$. By Proposition 13, $\mathbf{S}^\bullet$ is a stable extension in $\mathcal{F}_\Sigma^{\mathbb{D}_2}$. Since $\mathbf{F}_\Sigma^{\mathbf{D}_{DD}}$ and $\mathcal{F}_\Sigma^{\mathbb{D}_2}$ are X-extensionally equivalent, $(\mathbf{S}^\bullet)_\bullet$ is a stable extension in $\mathbf{F}_\Sigma^{\mathbf{D}_{DD}}$. By Lemma 6, $\mathbf{S} = (\mathbf{S}^\bullet)_\bullet$. It follows that $\mathbf{S}$ is a stable extension in $\mathbf{F}_\Sigma^{\mathbf{D}_{DD}}$. Other cases can be proved similarly. □

The following proposition shows that the grounded extension, ideal extensions and eager extensions are identical in $\mathbf{F}_\Sigma^{\mathbf{D}_{DD}}$ and $\mathbf{F}_\Sigma^{\mathbf{D}_{DU}}$. It is proved with Proposition 15 for $\mathcal{F}_\Sigma^{\mathbb{D}_2}$ via extensional equivalence.

Proposition 18. *Let $(\mathcal{L}, \vdash)$ be an abstract logic, Σ a knowledge base. Let $\mathbf{S} \subseteq Ar_p(\Sigma)$. Then the following statements are equivalent:*

1. *$\mathbf{S}$ is the grounded extension in $\mathbf{F}_\Sigma^{\mathbf{D}_{DD}}/\mathbf{F}_\Sigma^{\mathbf{D}_{DU}}$.*
2. *$\mathbf{S}$ is an ideal extension in $\mathbf{F}_\Sigma^{\mathbf{D}_{DD}}/\mathbf{F}_\Sigma^{\mathbf{D}_{DU}}$.*
3. *$\mathbf{S}$ is an eager extension in $\mathbf{F}_\Sigma^{\mathbf{D}_{DD}}/\mathbf{F}_\Sigma^{\mathbf{D}_{DU}}$.*

Proof. The proof is similar to that of Proposition 17. Proposition 15 is used in place of Proposition 13. □

9. Related works and conclusion

This paper contributes to structured argumentation. First, we compare base argumentation and premise-conclusion argumentation in terms of argument evaluation. To this aim, we define the notion of extensional equivalence between base argumentation and premise-conclusion argumentation and define the notion of bisimulation between a base argumentation framework and a premise-conclusion argumentation framework. We show that the following BAFs and PAFs are bisimilar: $\mathcal{F}_\Sigma^{\mathbb{D}_1}$ and $\mathbf{F}_\Sigma^{\mathbf{D}_D}$, $\mathcal{F}_\Sigma^{\mathbb{D}_2}$ and $\mathbf{F}_\Sigma^{\mathbf{D}_{DD}}$, $\mathcal{F}_\Sigma^{\mathbb{D}_2}$ and $\mathbf{F}_\Sigma^{\mathbf{D}_{DU}}$, $\mathcal{F}_\Sigma^{\mathbb{D}_3}$ and $\mathbf{F}_\Sigma^{\mathbf{D}_U}$. We show that bisimulation implies extensional equivalence. This means that base argumentation is as good as premise-conclusion argumentation under certain conditions. With its obvious simplicity, base argumentation may be preferred in some cases.

Second, to illustrate how base argumentation, bisimulation and extensional equivalence can help to deepen our understanding of premise-conclusion argumentation, we investigate the extensional properties of a base argumentation framework $\mathcal{F}_\Sigma^{\mathbb{D}_2}$ and export them to premise-conclusion argumentation frameworks $\mathbf{F}_{\Sigma}^{\mathbf{D}_{DD}}$ and $\mathbf{F}_{\Sigma}^{\mathbf{D}_{DU}}$ via bisimulation and extensional equivalence. We show that there are essentially three kinds of extensions in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$, which are complete, preferred and grounded extensions, though seven kinds of extensions are considered in this paper and that the same results hold in $\mathbf{F}_{\Sigma}^{\mathbf{D}_{DD}}$ and $\mathbf{F}_{\Sigma}^{\mathbf{D}_{DU}}$. This result contributes to the question of whether the various extensional semantics are essentially different if arguments and attack relations are defined in terms of deductive logics. The new results about premise-conclusion argumentation are (1) there are essentially three kinds of extensions in $\mathbf{F}_{\Sigma}^{\mathbf{D}_{DD}}$ and $\mathbf{F}_{\Sigma}^{\mathbf{D}_{DU}}$ and (2) the extensions of $\mathbf{F}_{\Sigma}^{\mathbf{D}_{DD}}$ and $\mathbf{F}_{\Sigma}^{\mathbf{D}_{DU}}$ are identical, though their attack relations are distinct.

Now we discuss our results in the context of the literature.

Base argumentation and inconsistency handling The main question in *inconsistency handling* is what can be meaningfully inferred from an inconsistent knowledge base. Base argumentation is a system for inconsistency handling since it selects acceptable sets of subsets of the knowledge base through extensional semantics in abstract argumentation [15,31,10,16,11].

There are two types of non-inconsistency-tolerant methods discussed in the literature.[4]

The first type of methods is based on the notion of maximal consistent sets. Rescher and Manor [26] mentioned that a formula is accepted as a consequence of a knowledge base when it can be classically inferred from all maximal consistent subsets of the knowledge base (this is the so-called *universal consequence*) or from at least one maximal consistent subset (this is the so-called *existential consequence*). They pointed out that the first is too narrow, and the second is too broad and may include mutually inconsistent propositions. For this, they discussed *preferential consequences* defined using preference criteria.

The second type of methods is based on argumentation. Elvang-Göransson et al. [17] conceived arguments as premise-conclusion pairs (Γ, φ) where Γ is a subset of the possibly inconsistent knowledge base and there exists a natural-deduction proof of φ from Γ. It classifies arguments into five classes of degrees of acceptability. Benferhat et al. [4] refined the notion of arguments requiring that the set of premises is consistent and subset-minimal, i.e., Γ is a minimal subset of the knowledge base that logically deduces φ. Such arguments are called *premise-conclusion arguments* in this paper. Benferhat et al. [4] also suggested that a proposition can be inferred from an inconsistent knowledge base if the knowledge base contains a deductive argument that supports this proposition, but no deductive argument that supports its negation. Another argumentative proposal was given in Besnard and Hunter [6]. Apart from deductive arguments, it defines various kinds of counter-arguments and then formalizes the notion of argument structures which exhaustively collate arguments and counter-arguments. Argument structures are evaluated through *aggregation functions*. The last kind of argumentative methods is based on abstract argumentation frameworks (AFs). Perhaps the first paper to consider logical instantiations of AFs was Cayrol [12] which instantiates Dung's proposal with deductive arguments based on classical logic. Gorogiannis and Hunter [20] also instantiated abstract argumentation with premise-conclusion arguments and investigated postulates for attack relations and extensions. We refer the readers to Section 2.3 of Prakken [25] for a general overview of argumentation-based inference and Arieli et al. [2] for a survey on logic-based approaches to formal argumentation.

Difference between various extensional semantics Whether the various extensional semantics are essentially different if arguments and attack relations are defined in terms of deductive logics is an important question in the literature. This paper shows that there are essentially three kinds of extensions in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$: complete, grounded and preferred extensions. Grounded and preferred extensions are closely related to maximal consistent subsets of the knowledge base. These results also hold for premise-conclusion argumentation frameworks $\mathbf{F}_{\Sigma}^{\mathbf{D}_{DD}}$ and $\mathbf{F}_{\Sigma}^{\mathbf{D}_{DU}}$ by extensional equivalence. There are similar results about premise-conclusion argumentation in the literature.

Cayrol [12] showed that if direct undercut (i.e., $\mathbf{D}_{DU}$) is used, then stable extensions of an argumentation system correspond exactly to maximal (for set inclusion) consistent subsets of the knowledge base. Vesic and van der Torre [32] identified four conditions describing a class of attack relations which return extensions corresponding exactly to the maximal (for set inclusion) consistent subsets of the knowledge base and showed that it is possible to obtain a meaningful result which does not correspond to the maximal consistent subsets of the knowledge base. Amgoud and Besnard [1] showed that for an argumentation system that satisfies consistency and closure under sub-arguments and a *conflict-dependent*[5] attack relation, and any preferred extension $\mathbf{S}$, there exists $\Gamma \in \mathbf{MC}(\Sigma)$ such that $\mathrm{S}(\mathbf{S}) = \Gamma$.

Base argumentation and assumption-based argumentation Assumption-based argumentation (ABA) [29,14,3,22] is a structural argumentation system.

According to [22], an ABA framework consists of a deductive system, a non-empty set of assumptions and a contrary function $\overline{}$ mapping assumptions to a formula in the language. A *tree-based argument* in ABA is of the form $S \vdash_R \varphi$, where S is the set of premises, φ is the conclusion and R is the set of rules leading from S to φ. From a tree-based argument $S \vdash_R \varphi$, one can obtain a *core argument* (S, φ). A set of assumptions A attacks a set of assumptions B if $A \vdash_R \overline{\psi}$ for some $\psi \in B$, or equivalently, if there is a core argument $(A, \overline{\psi})$ for some $\psi \in B$. ABA selects acceptable subsets of the assumptions through argumentative semantics. Evaluations through tree-based arguments and core arguments are semantically equivalent [22].

[4] For a review of inconsistency-tolerant methods in inconsistency handling, see e.g., [5].

[5] An attack relation $\mathcal{R}_d$ is *conflict-dependent* if $((\Gamma, \varphi), (\Delta, \psi)) \in \mathcal{R}_d$ implies $\Gamma \cup \Delta$ is inconsistent.

Extensional equivalence (Definition 12) between a base argumentation framework and a premise-conclusion framework is similar to Theorem 4.3 in [14], which is about the correspondence between ABA frameworks and the abstract argumentation frameworks induced by ABA frameworks.

Moreover, the base argumentation framework $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$ can be seen as an instance of ABA by setting the deductive system to be the abstract logic considered in this paper, the set of assumptions to be the knowledge base in base argumentation, and the contrary function to be the negation in abstract logic. Recall that the attack relation $\mathbb{D}_2$ is defined as follows: Γ $\mathbb{D}_2$-attacks Δ if there exists $\psi \in \Delta$ such that $\Gamma \vdash \neg\psi$.

There are two differences. First, acceptable candidates in ABA are conflict-free subsets of the assumptions, while those in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$ are base arguments, each of which is a finite consistent subset of the assumptions and does not have a logically equivalent proper subset. The notion of being conflict-free in ABA is equivalent to the notion of consistency when we see $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$ as an instance of ABA. Second, ABA selects acceptable subsets of the assumptions, while $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$ selects acceptable sets of subsets of the assumptions. Such a difference can be neutralized by noticing that each complete extension in $\mathcal{F}_{\Sigma}^{\mathbb{D}_2}$ is of the form $\rho(\Gamma)$, where $\rho(\Gamma) = \{\Gamma' \subseteq \Gamma \mid \Gamma'$ is a base argument$\}$, and viewing Γ as the representative element.

Future work Future work includes base argumentation for non-flat knowledge base, i.e., knowledge base with priority/preference/probability, and base argumentation with concrete formal logics, like fuzzy logic, relevant logic, probability logic, etc.

CRediT authorship contribution statement

Jinsheng Chen: Writing – review & editing, Writing – original draft, Funding acquisition, Formal analysis, Conceptualization. **Beishui Liao:** Funding acquisition, Conceptualization. **Leendert van der Torre:** Funding acquisition, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

The authors would like to thank the anonymous reviewers for making a number of important suggestions for improving the paper. Jinsheng Chen acknowledges financial support from the MOE (Ministry of Education in China) Project of Humanities and Social Sciences (Project No. 23YJC72040001). Beishui Liao acknowledges financial support from the National Social Science Foundation Major Project of China under grants No.20&ZD047 and the National Key Research and Development Program of China under grant No. 2022YFC3340900. Leon van der Torre acknowledges financial support from the Luxembourg National Research Fund (FNR), grant reference (O20/14776480/DELIGHT, INTER/DFG/23/17415164/LoDEx, C22/SC/17111440/EAI, INTER/AUDACE/21/16695098/INTEGRAUTO, INTER/CHIST/19/14589586/EXPECTATION).

References

[1] L. Amgoud, P. Besnard, Logical limits of abstract argumentation frameworks, J. Appl. Non-Class. Log. 23 (3) (2013) 229–267.
[2] O. Arieli, A. Borg, J. Heyninck, C. Straßer, Logic-based approaches to formal argumentation, J. Appl. Log. 2631 (6) (2021) 1793.
[3] Z. Bao, K. Čyras, F. Toni, ABAplus: attack reversal in abstract and structured argumentation with preferences, in: International Conference on Principles and Practice of Multi-Agent Systems, Springer, 2017, pp. 420–437.
[4] S. Benferhat, D. Dubois, H. Prade, Argumentative inference in uncertain and inconsistent knowledge bases, in: Uncertainty in Artificial Intelligence, Elsevier, 1993, pp. 411–419.
[5] L. Bertossi, A. Hunter, T. Schaub, Introduction to inconsistency tolerance, in: Inconsistency Tolerance, Springer, 2005, pp. 1–14.
[6] P. Besnard, A. Hunter, A logic-based theory of deductive arguments, Artif. Intell. 128 (1–2) (2001) 203–235.
[7] P. Besnard, A. Hunter, Constructing argument graphs with deductive arguments: a tutorial, Argument Comput. 5 (1) (2014) 5–30.
[8] P. Besnard, A. Hunter, A review of argumentation based on deductive arguments, in: Handbook of Formal Argumentation, College Publications, 2018, pp. 437–484, chap. 9.
[9] P. Blackburn, M. De Rijke, Y. Venema, Modal Logic, vol. 53, Cambridge University Press, 2002.
[10] M. Caminada, Semi-stable semantics, in: Proceedings of the 1st International Conference on Computational Models of Argument (COMMA 2006), IOS Press, 2006, pp. 121–130.
[11] M. Caminada, Comparing two unique extension semantics for formal argumentation: ideal and eager, in: Proceedings of the 19th Belgian-Dutch Conference on Artificial Intelligence (BNAIC 2007), Utrecht University Press, 2007, pp. 81–87.
[12] C. Cayrol, On the relation between argumentation and non-monotonic coherence-based entailment, in: Proceedings of 14th International Joint Conference on Artificial Intelligence (IJCAI 1995), 1995, pp. 1443–1448.
[13] J. Chen, B. Liao, L.v.d. Torre, Base argumentation as an abstraction of deductive argumentation, in: International Conference on Logic and Argumentation, Springer, 2021, pp. 468–476.

[14] K. Čyras, X. Fan, C. Schulz, F. Toni, Assumption-based argumentation: disputes, explanations, preferences, in: Handbook of Formal Argumentation, College Publications, 2018, pp. 365–408, chap. 7.
[15] P.M. Dung, On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games, Artif. Intell. 77 (2) (1995) 321–357.
[16] P.M. Dung, P. Mancarella, F. Toni, Computing ideal sceptical argumentation, Artif. Intell. 171 (10–15) (2007) 642–674.
[17] M. Elvang-Göransson, P.J. Krause, J. Fox, Acceptability of arguments as 'logical uncertainty', in: European Conference on Symbolic and Quantitative Approaches to Reasoning and Uncertainty, Springer, 1993, pp. 85–90.
[18] A.J. García, G.R. Simari, Defeasible logic programming: DeLP-servers, contextual queries, and explanations for answers, Argument Comput. 5 (1) (2014) 63–88.
[19] A.J. García, G.R. Simari, Argumentation based on logic programming, in: Handbook of Formal Argumentation, College Publications, 2018, pp. 409–436, chap. 8.
[20] N. Gorogiannis, A. Hunter, Instantiating abstract argumentation with classical logic arguments: postulates and properties, Artif. Intell. 175 (9–10) (2011) 1479–1497.
[21] R. Gorrieri, C. Versari, Introduction to Concurrency Theory: Transition Systems and CCS, Springer, 2015.
[22] T. Lehtonen, A. Rapberger, M. Ulbricht, J.P. Wallner, Argumentation frameworks induced by assumption-based argumentation: relating size and complexity, in: Proceedings of the International Conference on Principles of Knowledge Representation and Reasoning, vol. 19, 2023, pp. 440–450.
[23] S. Modgil, H. Prakken, The ASPIC+ framework for structured argumentation: a tutorial, Argument Comput. 5 (1) (2014) 31–62.
[24] S. Modgil, H. Prakken, Abstract rule-based argumentation, in: Handbook of Formal Argumentation, College Publications, 2018, pp. 287–364, chap. 6.
[25] H. Prakken, Historical overview of formal argumentation, in: Handbook of Formal Argumentation, College Publications, 2018, pp. 75–144, chap. 2.
[26] N. Rescher, R. Manor, On inference from inconsistent premisses, Theory Decis. 1 (2) (1970) 179–217.
[27] G. Rosen, A. Byrne, J. Cohen, S.V. Shiffrin, The Norton Introduction to Philosophy, WW Norton & Company, 2015.
[28] A. Tarski, On Some Fundamental Concepts of Metamathematics, Oxford, 1956.
[29] F. Toni, A tutorial on assumption-based argumentation, Argument Comput. 5 (1) (2014) 89–117.
[30] M.Y. Vardi, On epistemic logic and logical omniscience, in: Theoretical Aspects of Reasoning About Knowledge, Elsevier, 1986, pp. 293–305.
[31] B. Verheij, Two approaches to dialectical argumentation: admissible sets and argumentation stages, in: Proc. NAIC, vol. 96, 1996, pp. 357–368.
[32] S. Vesic, L.W.N. van der Torre, Beyond maxi-consistent argumentation operators, in: L.F. del Cerro, A. Herzig, J. Mengin (Eds.), Proceedings of JELIA 2012, in: Lecture Notes in Computer Science, vol. 7519, Springer, 2012, pp. 424–436.