WEAKLY GIBBSIAN REPRESENTATIONS FOR JOINT MEASURES OF QUENCHED LATTICE SPIN MODELS *

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Abstract: Can the joint measures of quenched disordered lattice spin models (with finite range) on the product of spin-space and disorder-space be represented as (suitably generalized) Gibbs measures of an "annealed system"? - We prove that there is always a potential (depending on both spin and disorder variables) that converges absolutely on a set of full measure w.r.t. the joint measure ("weak Gibbsianness"). This "positive" result is surprising when contrasted with the results of a previous paper [K6], where we investigated the measure of the set of discontinuity points of the conditional expectations (investigation of "a.s. Gibbsianness"). In particular we gave natural "negative" examples where this set is even of measure one (including the random field Ising model).

Further we discuss conditions giving the convergence of vacuum potentials and conditions for the decay of the joint potential in terms of the decay of the disorder average over certain quenched correlations. We apply them to various examples. From this one typically expects the existence of a potential that decays superpolynomially outside a set of measure zero. Our proof uses a martingale argument that allows to cut (an infinite-volume analogue of) the quenched free energy into local pieces, along with generalizations of Kozlov's constructions.

Key Words: disordered Systems, Gibbs-measures, non-Gibbsian measures, Ising model, random field model, random bond model, dilute Ising model

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I. Introduction

Consider the joint measure corresponding to a random infinite-volume Gibbs measure of a disordered lattice spin system. By this we mean the measure $I\!P(d\eta)\mu[\eta](d\sigma)$ on the product space of disorder variables η and spin variables σ. Here $\mu[\eta](d\sigma)$ is a random Gibbs measure and $I\!P$ is the a-priori distribution of the disorder variables. Prototypical examples for such quenched random systems are the random field Ising model or an Ising model with random couplings.

In this paper we investigate the question: When can these measures be understood as Gibbs measures on the skew space, respectively suitable generalizations thereof? More specifically, are there well-defined Hamiltonians, given in terms of interaction potentials depending on both spin and disorder variables, that provide an annealed description for such a system? The formal description of disordered systems in terms of such potentials was termed "Morita's equilibrium ensemble approach to disordered systems" (see e.g [Ku1,2], [MKu], [Mo], [SW] and references in [Ku2]) in the theoretical physics community. However, the existence of such Hamiltonians was never investigated rigorously but taken for granted, and various approximation schemes were based on the truncation of the corresponding potentials. In this respect there is an analogy between the problems of the existence of joint potentials and of the existence of "renormalized potentials" that are supposed to give a Gibbsian description of a measure that appears as an image measure of a Gibbs measure under a renormalization group transformation. There is a huge literature about the latter ones but the present question has remained mathematically neglected until recently ([EMSS], [K6]).

Now, mathematically, it turns out that the answer to our question is a somewhat complicated but interesting one. It depends on the kind of generalization of the notion of Gibbsianness one is asking for and on the specific system. Therefore such joint measures corresponding to quenched random systems provide a rich class of examples to illustrate the subtleties of the different generalizations of the notion of Gibbsianity. We believe that, while interesting in itself, the study of these measures is also valuable for the understanding of the fine (and not always very intuitive) distinctions that are necessary if one attempts to extend Gibbsian theory to non-Gibbsian measures.

Recall that Gibbs measures of an infinite-volume lattice system are characterized by the fact that their conditional expectations (given the values of the variables outside of a finite volume) can be written in terms of an absolutely convergent interaction potential. Equivalently, they are the measures for which these conditional expectations are continuous functions of the conditioning. (The less trivial part of the equivalence, i.e. existence of a potential assuming continuity of conditional expectations, is due to the construction of [Koz]). For general information about scenarios of the failure of the Gibbsian property for lattice measures and possible generalizations of Gibbsianness see e.g. [F],[E],[DS],[BKL],[MRM], [MRSM], references therein, and the basic paper [EFS].

In the first mathematical paper [EMSS] which studied a joint measure of a quenched random system it was shown that the joint measure resulting from the diluted Ising ferromagnet at low temperatures is not a Gibbs measure in the strict sense described above: [EMSS] showed that there is a point of essential discontinuity in the conditional expectations as a function of the conditioning. So, the measure does not allow for a Hamiltonian constructed from an absolutely summable interaction potential. However, the set of such discontinuities has zero measure in this example. Measures with this property are commonly called "almost Gibbsian" measures. The notion of "almost Gibbsianness" is a straightforward measure-theoretic attempt to generalize the classical notion of Gibbsianness where the conditional expectations are continuous everywhere.

In a recent paper [K6] we investigated the question of discontinuity of the conditional expectations in the general setup of quenched lattice spin systems with finite range quenched Hamiltonians depending on independent disorder variables. In particular, we gave an example where the set of discontinuities was even a **full measure set**. So, even worse, this measure even fails to be "almost Gibbsian"! The example was the random field Ising model in the phase transition regime. It is particularly illuminating because it shows in a transparent manner a more general fact: The question of discontinuity of the conditional expectations is related to whether a discontinuity can be felt on certain local expectations of the quenched measure by varying the disorder variables arbitrarily far away. The local expectation under consideration is just the magnetization for the random field Ising model; more generally this has to be replaced by the spin-observable conjugate to the independent disorder variables. In [K6] we also discussed another interesting phenomenon: We argued that whether the set of discontinuity points is of measure zero or one can depend on the random Gibbs measure, for the same choice of the parameters. This phenomenon should appear in the random bond ferromagnet at low temperatures, weak disorder, and high dimensions: We argued that it is to be expected that the set of discontinuities should be of measure zero for the ferromagnetic plus state while it should be of measure one for the random Dobrushin state.

While we focused on "almost Gibbsianness" in [K6], the aim of the present paper is to find out what can be said about "weak Gibbsianness". The latter notion is a different attempt to weaken (even more) the classical notion of Gibbs measure. Here one requires only the existence of a potential that is convergent (or even absolutely convergent) on a full measure set (and not necessarily everywhere). [MRM] noted that, in general, an almost Gibbsian measure always has a potential that is convergent on a set of full measure. It is however **not** expected that there is always an **absolutely** convergent potential in this situation. Also, [MRM] gave an example of a measure having a convergent potential which was not almost Gibbsian.

In this note we will give a completely general positive answer to the question of weak Gibbsianness for our measures. That is, at least from the point of view of weak Gibbsianness, the situation gets easier again. We will show:

The joint measures corresponding to a random infinite-volume Gibbs measure always possess a potential that converges absolutely on a full measure set.

For the specific example of the random field Ising model in the phase transition regime this gives, together with the result of [K6] the following interesting statement: The set of discontinuity points of the joint measure has full measure, but still there is a potential that converges absolutely on a set of full measure. [1] So, almost Gibbsianness does not hold, but weak Gibbsianness does (even in a strong form). In fact, we expect the convergence to be very fast on a set of measure one (see Chapter V.)

Our existence result is true for any quenched lattice spin systems with finite range quenched Hamiltonians depending on sitewise independent disorder variables. Observe that e.g. models with independent bond disorder are also included. (Indeed, each bond on the lattice can be accociated to one of its vertices according to some fixed prescription; then the 'site variable' at a given site is formed by the collection of all bond variables that are associated to this site.)

We stress here that **no continuity assumptions** at all are needed on the measures involved. This may seem surprising and is a main non-trivial point. Let us describe our results at first in words, before we put them down in precise formulas. They will all have the following form: We construct a potential and explain its properties and how it is related to the given "quenched potential" that is the starting point and defines the system we are dealing with.

Now, to put the first result in perspective, we remark that in the case of a general lattice measure, the existence of an a.s. convergent potential can be obtained once there is at least one direction of (a.s.) continuity for the conditional expectations (see [MRM]) using the corresponding vacuum potential. Due to the special form of the joint measures we are considering here, we can improve on this in our case (see Theorems 2.1,2.3). For this we take advantage of the specific form of the infinite-volume conditional expectations of the joint measures derived in Chapter II. The trick to get the stronger result is to use not a vacuum potential, but a different one; this will allow to conclude convergence of the potential by a soft martingale argument. From this we can get an existence result for an a.s. **absolutely** convergent potential generalizing the one of

[1] Recently [Le] constructed an independent example of a lattice measure (not related to random systems) to illustrate that this phenomenon can really occur.

[Koz]. We remark that also for this latter step we are again exploiting the special nature of our measures; it would not work for a general lattice measure.

Nevertheless, it is also interesting to see what can be said about the convergence of vacuum potentials (see Theorem 2.2). For this we need in fact some continuity, conveniently expressed in terms of the behavior of the corresponding infinite-volume Gibbs state: One needs continuity of the corresponding infinite-volume quenched Gibbs-expectation of the spin-observable conjugate to the independent disorder variables, as a function of the quenched variables, in the direction of a certain realization of the disorder. These are the same observables whose behavior was crucial also for the question of "almost sure Gibbsianness".

Next, if one would like to have more information about the decay of the potential, one has to assume some information about the clustering properties of the quenched random system. We relate the decay of a joint potential to the decay of disorder-averages of certain quenched correlations in Theorem 2.4. These correlations are taken between the spin- observables conjugate to the independent disorder variables, the same ones as above. Physically, superpolynomial decay of such averaged correlations is typically to be expected (off the critical point). So, we should typically expect the existence of a potential that decays superpolynomially outside of a set of measure zero. Of course, to prove it, specific analysis of the system under consideration is needed, which can be very hard.

The paper is organized as follows. In Chapter II we define the class of models we will treat and state our results in precise terms. In Chapter III we prove the important formula for the infinite-volume conditional expectations of the joint measure that is the starting point of the following. In Chapter IV we will prove the theorems stated in Chapter II. In Chapter V we will discuss the examples of the random field Ising model, Ising models with random couplings, and the diluted Ising ferromagnet, including some heuristic considerations.

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II. The Models and the Results

Denote by $\Omega = \Omega_0^{\mathbb{Z}^d}$ the space of **spin-configurations** $\sigma = (\sigma_x)_{x\in\mathbb{Z}^d}$, where Ω_0 is a finite set. Similarly we denote by $\mathcal{H} = \mathcal{H}_0^{\mathbb{Z}^d}$ the space of **disorder variables** $\eta = (\eta_x)_{x\in\mathbb{Z}^d}$ entering the model, where $\mathcal{H}_0$ is a finite set. Each copy of $\mathcal{H}_0$ carries a measure $\nu(d\eta_x)$ and $\mathcal{H}$ carries the product-measure over the sites, $I\!P = \nu^{\otimes_{\mathbb{Z}^d}}$. We denote the corresponding expectation by

$I\!E$. The space of joint configurations $\bar\Omega := \Omega\times\mathcal{H} = (\Omega_0\times\mathcal{H}_0)^{\mathbb{Z}^d}$ is called **skew space**. It is equipped with the product topology and the corresponding Borel sigma algebra.

A **potential on the joint variables** is a family U of real functions $U_A : \bar\Omega \to I\!R$ where A runs over the finite subsets of $\mathbb{Z}^d$ s.t. $U_A(\xi)$ depends only on ξ_A. We consider disordered models whose finite volume Gibbs-measures can be written in terms of a potential $\Phi = (\Phi_A)_{A\subset\mathbb{Z}^d}$ on the joint variables. In this context we will call Φ the **disordered potential** (or **definining potential**). We fix a realization of the disorder η and define probability measures $\mu_\Lambda^{\sigma^{\text{b.c.}}}[\eta]$ on the spin space Ω, called the **quenched finite volume Gibbs measures**, by

$$\mu_\Lambda^{\sigma^{\text{b.c.}}}[\eta](\sigma) := \frac{e^{-\sum_{A:A\cap\Lambda\neq\emptyset}\Phi_A(\sigma_\Lambda\sigma^{\text{b.c.}}_{\mathbb{Z}^d\setminus\Lambda},\eta)}}{\sum_{\tilde\sigma_\Lambda} e^{-\sum_{A:A\cap\Lambda\neq\emptyset}\Phi_A(\tilde\sigma_\Lambda\sigma^{\text{b.c.}}_{\mathbb{Z}^d\setminus\Lambda},\eta)}} 1_{\sigma_{\mathbb{Z}^d\setminus\Lambda}=\sigma^{\text{b.c.}}_{\mathbb{Z}^d\setminus\Lambda}} \tag{2.1}$$

The finite-volume summation is over $\sigma_\Lambda\in\Omega_0^\Lambda$. The symbol $\sigma_\Lambda\sigma^{\text{b.c.}}_{\mathbb{Z}^d\setminus\Lambda}$ denotes the configuration in Ω that is given by σ_x for $x\in\Lambda$ and by $\sigma_x^{\text{b.c.}}$ for $x\in\mathbb{Z}^d\setminus\Lambda$. We assume for simplicity **finite range**, i.e. that $\Phi_A=0$ for diam$A>r$. This form is really quite general. It is a simple matter to write the random field Ising model or the random bond Ising model in the above form.

Next, we suppose from the beginning that we have the existence of a weak limit

$$\lim_{\Lambda\uparrow\mathbb{Z}^d}\mu_\Lambda^{\sigma^{\text{b.c.}}_{\partial\Lambda}}[\eta] = \mu[\eta] \tag{2.2}$$

for $I\!P$-a.e. $\eta\equiv\eta_{\mathbb{Z}^d}$ with a nonrandom boundary condition $\sigma^{\text{b.c.}}$. In ferromagnetic examples like the random field Ising model this can be concluded by monotonicity arguments. Note that there is however no general argument that would give the existence of this limit - indeed it is expected to fail e.g. for low temperature spinglasses.[2]

Assuming (2.2) it follows that $\mu_\infty[\eta_{\mathbb{Z}^d}]$ is an infinite-volume Gibbs measure for P-a.e. η that depends measurably on η. We look at spins and disorder variables at the same time and define **joint spin variables** $\xi_x = (\sigma_x,\eta_x)\in\Omega_0\times\mathcal{H}_0$. The central object of our study is the corresponding **infinite-volume joint measure** on the skew space $(\Omega_0\times\mathcal{H}_0)^{\mathbb{Z}^d}$ defined by

$$K(d\sigma,d\eta) := I\!P(d\eta)\mu[\eta](d\sigma) \tag{2.3}$$

[2] Side-remark about the relation to "metastates": It is this existence problem that led to the introduction of the general notion of metastates, which are distributions of Gibbs-measures, see e.g. [NS1]-[NS5], [K2]-[K5]. Also, more generally than in the present note, in large parts of [K6] we did not assume the a.s. convergence of the random finite volume Gibbs measures, but only the weaker property of convergence of the corresponding finite volume joint measures. Assuming the existence of a corresponding metastate, such a measure K is its barycenter. The case of the present note corresponds to the trivial metastate which is supported only on a single state $\mu[\eta]$.

We say that a potential U on the joint variables is a **potential for the joint measure** K if U produces the correct conditional expectations for K, i.e.

$$\frac{e^{-\sum_{A:A\cap\Lambda\neq\emptyset}U_A(\xi)}}{\sum_{\tilde\xi_\Lambda}e^{-\sum_{A:A\cap\Lambda\neq\emptyset}U_A(\tilde\xi_\Lambda\xi_{\mathbb{Z}^d\setminus\Lambda})}}=K[\xi_\Lambda|\xi_{\mathbb{Z}^d\setminus\Lambda}] \tag{2.4}$$

for K-a.e. ξ. This work is about the existence of such a potential. It provides a description of the joint measure as an "annealed system". This notion should not be confused with the following one.

We call a potential U^{triv} on the joint variables a **trivial annealed potential contribution** if it is finite range and produces the "trivial annealed local specification", i.e.

$$\frac{e^{-\sum_{A\cap\Lambda\neq\emptyset}U_A^{ann}(\sigma_\Lambda\sigma^{\text{b.c.}}_{\mathbb{Z}^d\setminus\Lambda},\eta_\Lambda\eta^{\text{b.c.}}_{\mathbb{Z}^d\setminus\Lambda})}}{\sum_{\tilde\sigma_\Lambda,\tilde\eta_\Lambda}e^{-\sum_{A\cap\Lambda\neq\emptyset}U_A^{triv}(\tilde\sigma_\Lambda\sigma^{\text{b.c.}}_{\mathbb{Z}^d\setminus\Lambda},\tilde\eta_\Lambda\eta^{\text{b.c.}}_{\mathbb{Z}^d\setminus\Lambda})}}=\frac{\nu(\eta_\Lambda)e^{-\sum_{A\cap\Lambda\neq\emptyset}\Phi_A(\sigma_\Lambda\sigma^{\text{b.c.}}_{\mathbb{Z}^d\setminus\Lambda},\eta_\Lambda\eta^{\text{b.c.}}_{\mathbb{Z}^d\setminus\Lambda})}}{\sum_{\tilde\sigma_\Lambda,\tilde\eta_\Lambda}\nu(\tilde\eta_\Lambda)e^{-\sum_{A\cap\Lambda\neq\emptyset}\Phi_A(\tilde\sigma_\Lambda\sigma^{\text{b.c.}}_{\mathbb{Z}^d\setminus\Lambda},\tilde\eta_\Lambda\eta^{\text{b.c.}}_{\mathbb{Z}^d\setminus\Lambda})}} \tag{2.5}$$

One such potential is $U_A^{triv}(\sigma,\eta)=\Phi_A(\sigma,\eta)-1_{A=\{x\}}\log\nu(\eta_x)$; evidently it is unique only up to "physical equivalence". We remark that, of course, the problem of classifying the equivalent potentials U for given ν,Φ is long solved and can be found in [Geo], see paragraphs (2.3) and (2.4) therein. We call the local specification on the r.h.s. "annealed" too because it describes a joint system given by an Hamiltonian which is simply the quenched Hamiltonian and the logarithm of the a priori measure given by the independent distribution $\mathbb{P}$ for the disorder variables. Of course, its properties may differ completely from the quenched system. The reason for the introduction of U^{triv} and its name is that it will appear as a 'trivial' part of the potentials of the joint measures that we are looking for.

Finally, a potential U is called **summable for** ξ if, for any $\Lambda\subset\mathbb{Z}^d$, we have that the limit $\lim_{\Delta\uparrow\mathbb{Z}^d}\sum_{A:A\cap\Lambda\neq\emptyset,A\subset\Delta}U_A(\xi)=:\sum_{A:A\cap\Lambda\neq\emptyset}U_A(\xi)$ exists and is independent of the sequence of Δ's. This is needed for the sums in (2.4) to make sense. U is called **absolutely summable for** ξ if, for any $\Lambda\subset\mathbb{Z}^d$ we have that $\sup_{\Delta\subset\mathbb{Z}^d}\sum_{A:A\cap\Lambda\neq\emptyset,A\subset\Delta}|U_A(\xi)|<\infty$.

Now, the most natural approach to find a potential for the joint measure is to write down a formal vacuum potential on the joint space and ask what we can say about its convergence (see Theorem 2.2). We remind the reader that a potential U is called vacuum potential with vacuum $\hat\xi$, if $U_A(\xi_{A\setminus x}\hat\xi_x)=0$ whenever $x\in A$. However, it turns out that we get our strongest general existence result of Theorem 2.1 for a different potential. To this end, let $\alpha(d\xi)$ be a product probability measure. Then, a potential U is called **α-normalized** if $\int\alpha_x(d\tilde\xi_x)U_A(\xi_{A\setminus x}\tilde\xi_x)=0$ whenever $x\in A$. Obviously, for $\alpha=\delta_{\hat\xi}$, an α-normalized potential is a vacuum potential with vacuum $\hat\xi$. This notion was first introduced by Israel [I] but we use the terminology of Georgii. α-normalized potentials (for suitably chosen α) can sometimes have better convergence properties compared to those of vacuum potentials also in the standard Gibbs formalism (see [Geo] example (2.38), [EFS] page 958).

In the following we assume that we are given a joint measure of the type (2.5) corresponding to a quenched random lattice model defined by (2.1), (2.2). Then the following statements hold.

Theorem 2.1 (Existence of a.s. summable potential): *There exists a potential U for K that is summable for K-a.e. ξ. This is true under no further assumptions on the continuity properties of $\mu[\eta]$. This potential has the form $U(\sigma,\eta) = U^{triv}(\sigma,\eta) + U^{\mathrm{fe}}_{\mu}(\eta)$. In this equation U^{triv} is an arbitrary representative of the "trivial annealed potential contribution" (defined up to physical equivalence).*

U^{fe}_{μ} is a potential depending only on η which is convergent for $\mathbb{P}$-a.e. η. As a potential on the disorder space it is $\mathbb{P}$-normalized. In general, two different measurable infinite-volume Gibbs-states $\mu : \eta \mapsto \mu[\eta]$ corresponding to the same random local specification will yield different U^{fe}_{μ}.

The notation $U^{\mathrm{fe}}_{\mu}(\eta)$ is meant to suggest to the reader, that this potential comes from a decomposition into local terms of what in finite volume would be the disorder dependent free energies of the quenched system. This will become clear in the proofs. An analogous finite volume quantity is called "disorder potential" in [Ku2].

To describe the kind of continuity we need for the existence of the vacuum potential in detail we need some more notation. For a subset $V \subset \mathbb{Z}^d$, we call the expression

$$\Delta H_V(\sigma_{\overline{V}}, \eta^1_V, \eta^2_V, \eta_{\partial V}) := \sum_{A: A\cap V \neq \emptyset} \Big(\Phi_A\left(\sigma_{\overline{V}}, \eta^1_V \eta_{\partial V}\right) - \Phi_A\left(\sigma_{\overline{V}}, \eta^2_V \eta_{\partial V}\right)\Big) \tag{2.6}$$

the V**-variation of the Hamiltonian w.r.t. the disorder variables**. To denote the corresponding function on the spin-variables obtained by fixing the disorder variables we will drop the spin-variable σ on the l.h.s. of (2.6). In particular, for $V = \{x\}$, the expression (2.6) is the observable conjugate to the independent disorder variable η_x. We put

$$Q_x(\eta^1_x, \eta^2_x, \eta_{\mathbb{Z}^d\setminus x}) := \mu[\eta^2_x, \eta_{\mathbb{Z}^d\setminus x}](e^{-\Delta H_x(\eta^1_x, \eta^2_x, \eta_{\partial x})}) \tag{2.7}$$

for its quenched expectation.

Theorem 2.2 (A.s. summability of vacuum potential): *Suppose moreover that there exists a direction $\hat{\eta}$ of a.s. continuity for the quenched expectation of the spin observable conjugate to the disorder variables, i.e.*

$$\lim_{\Lambda\uparrow\mathbb{Z}^d} Q_x(\eta^1_x, \eta^2_x, \eta_{\Lambda\setminus x}\hat{\eta}_{\mathbb{Z}^d\setminus\Lambda}) = Q_x(\eta^1_x, \eta^2_x, \eta_{\mathbb{Z}^d\setminus x}) \tag{2.8}$$

for all x, η_x^1, η_x^2, for $I\!P$-a.e. η. We assume that Q is defined by the weak limit (2.2) and (2.7) and this weak limit exists for $I\!P$-a.e. η. Here we have fixed a nonrandom boundary condition $\sigma^{\text{b.c.}}$ for those η that are **not** *in the $I\!P$-zero-set of η's of the form $(\eta_\Lambda \hat{\eta}_{\boldsymbol{Z}^d\setminus\Lambda})$. Moreover we assume that (2.2) also exists for $\hat{\eta}$ (and thus for all the countably many η's of the form 3.1 $(\eta_\Lambda \hat{\eta}_{\boldsymbol{Z}^d\setminus\Lambda})$), with some* **possibly different** *boundary condition $\hat{\sigma}^{\text{b.c.}}$.*

Then there is a vacuum potential $V_\mu^{\text{fe}}(\eta)$ on the disorder space with vacuum $\hat{\eta}$ s.t. $U'(\sigma,\eta) = U^{triv}(\sigma,\eta) + V_\mu^{\text{fe}}(\eta)$ is a potential for the joint measure K which is summable K-a.s.. Here U^{triv} is an arbitrary representative of the "trivial annealed potential contribution", as in Theorem 2.1.

Note that our hypothesis is weaker than requiring a.s. continuity of $\mu[\eta]$ itself in direction $\hat{\eta}$ (by which one understands continuity of **all** probabilities $\mu[\eta](\sigma_\Lambda)$ in this direction.) Note that, in general, the **same** choices of boundary conditions to construct the state $\mu[\hat{\eta}]$, and the state $\mu[\eta]$ for typical η might yield a state of **different** type. [In V(iii) (before the proof of Proposition 5.1 is given) we discuss the example of the dilute Ising model where a fixed "plus-minus" boundary condition $\sigma^{\text{b.c.}}$ is expected to produce a symmetric linear combination of the plus and the minus state for typical occupation numbers η, while it should produce an interface-state ("Dobrushin-state") for $\hat{\eta}$ describing the fully occupied system), in a certain parameter region.]

Now, in the situation of Theorem 2.2, fix any $\hat{\sigma}$. Then we can in particular choose $U^{triv}(\sigma,\eta)$ to be the unique vacuum potential for the annealed system with vacuum $(\hat{\sigma},\hat{\eta})$. [3] This gives the simple

Corollary 1: *If $\hat{\eta}$ is a direction of continuity for $\mu(\eta)$, for any $\hat{\sigma} \in \Omega$, the formal vacuum potential for K with vacuum $\hat{\xi} = (\hat{\sigma},\hat{\eta})$ is convergent for K-a.e ξ. Here we have assumed that $\mu[\eta]$ is defined by the weak limit (2.2) with boundary conditions as in the hypothesis of Theorem 2.2.*

Remark: If K is translation-invariant, so are the potentials constructed in the proof of Theorem 2.1 and Theorem 2.2. In general, they need not be absolutely summable.

The proof of Theorem 2.2 also gives

[3] A clear proof of the existence of an α-normalized convergent potential in the case of *continuous* conditional expectations can be found in [Geo] Theorem (2.30). Under our assumptions of discrete joint spin space and finite range of the defining disordered potential Φ this theorem shows in particular: For any α there exists a unique equivalent α-normalized potential for the annealed system with the same range.

Corollary 2: *The sum* $\sum_{A:A\cap\Lambda\neq\emptyset}\int I\!P(d\tilde\eta)V^{\text{fe}}_{\mu;A}(\tilde\eta)$ *converges. Hence* $U^{triv}_A(\sigma,\eta)+\Big[V^{\text{fe}}_{\mu;A}(\eta)-\int I\!P(d\tilde\eta)V^{\text{fe}}_{\mu;A}(\tilde\eta)\Big]$ *is a potential for the joint measure which is summable K-a.s., too.*[4]

From Theorem 2.1 one can obtain an absolutely summable potential, if one gives up translation invariance.

Theorem 2.3 (Existence of a.s. absolutely convergent potential): *There exists an* **a.s. absolutely summable** *potential* U^{abs} *for the joint measure* K *of the form* $U^{\text{abs}}(\sigma,\eta)=U^{triv}(\sigma,\eta)+U^{\text{fe, abs}}_{\mu}(\eta)$. *Here, as above,* U^{triv} *is an arbitrary representative of the "trivial annealed potential contribution".* $U^{\text{fe, abs}}_{\mu}$ *is a potential depending only on* η *which is* **absolutely** *convergent for* $I\!P$*-a.e.* η. $U^{\text{fe, abs}}_{\mu}$ *is not necessarily translation invariant even if* K *is translation invariant. As in Theorem 2.1, this results holds under no further continuity assumptions on* $\mu[\eta]$.

Remark: In fact the new 'free energy' potential $U^{\text{fe, abs}}_{\mu}$ is even integrable w.r.t. K (which is to say integrable w.r.t. $I\!P$). There is no estimate on the speed of convergence.

$U^{\text{fe, abs}}_{\mu}(\eta)$ is supported on a very sparse system of subsets of $\mathbb{Z}^d$. It is obtained by a resummation of the $I\!P$-normalized 'free energy' potential U^{fe}_{μ} from the construction Kozlov used on the vacuum potential in the case of a measure with continuous conditional expectations [Koz]. We remark that the same construction can in general **not** be applied to the vacuum potential V^{fe}_{μ} of Theorem 2.2, unless there is additional information on its decay.

Remark: Let us also comment on the easy case, when Q is continuous **everywhere**, by which we mean that

$$\lim_{\Lambda\uparrow\mathbb{Z}^d}\sup_{\hat\eta}\left|Q_x(\eta^1_x,\eta^2_x,\eta_{\Lambda\setminus x}\hat\eta_{\mathbb{Z}^d\setminus\Lambda})-Q_x(\eta^1_x,\eta^2_x,\eta_{\mathbb{Z}^d\setminus x})\right|=0 \tag{2.9}$$

for all η and all x, η^1_x, η^2_x. Then, the infinite volume conditional expectations of K are continuous, and so K is a Gibbs measure. The "free energy potentials" U^{fe}_{μ} (of Theorem 2.1) and V^{fe}_{μ} (of Theorem 2.2) are both convergent everywhere. Furthermore, the stronger version of Theorem 2.3 holds where "a.s. absolute summability" is strengthened to "absolute summability everywhere".

To get an absolutely summable potential for the *joint measure* that is also translation invariant, more information on the clustering properties of the quenched system on the average is needed. Theorem 2.4 below describes the existence of an a.s. absolutely summable potential

[4] This proves general existence of potentials of the form generalizing the one that was written down in finite volume in [Ku2 (32)] for the special case of the dilute Ising model, where no proof of the infinite-volume limit was given (see also Chapter V).

that is **translation invariant**, if the measure K is. Moreover it gives information about the decay of this potential.

Theorem 2.4 (A.s. absolutely summable translation invariant potential):

Assume that the averaged quenched correlations satisfy the decay property $\sum_{m=1}^{\infty} m^{2d-1}\bar c(m) < \infty$ *where* $\bar c(m) := \sup_{\substack{x,y:|x-y|=m\\ \eta_x,\eta_y\in\mathcal{H}_0}} \int I\!P(d\tilde\eta)\,|c_{x,y}(\eta_x,\eta_y,\tilde\eta)|$ *with*

$$
\begin{aligned}
&c_{x,y}(\eta_x,\eta_y,\tilde\eta)\\
&:= \mu[\tilde\eta]\left(e^{-\Delta H_{\{x,y\}}(\eta_{\{x,y\}},\tilde\eta_{\{x,y\}},\tilde\eta\big|_{\partial\{x,y\}})}\right) - \mu[\tilde\eta]\left(e^{-\Delta H_x(\eta_x,\tilde\eta_x,\tilde\eta\big|_{\partial x})}\right)\mu[\tilde\eta]\left(e^{-\Delta H_y(\eta_y,\tilde\eta_y,\tilde\eta\big|_{\partial y})}\right)
\end{aligned}
\tag{2.10}
$$

Then there is an a.s. absolutely summable potential $U_{\mu}^{\text{fe,abs,inv}}(\eta)$ *on the disorder space s.t.* $U^{abs,inv}(\sigma,\eta) = U^{triv}(\sigma,\eta) + U_{\mu}^{\text{fe,abs,inv}}(\eta)$ *is a potential for the joint measure* K. *Here, as above,* U^{triv} *is an arbitrary representative of the "trivial annealed potential contribution".*

If K *is translation invariant, then* $U_{\mu}^{\text{fe,abs,inv}}(\eta)$ *is translation invariant, too.*

Remark: Again, the potential is even integrable. Moreover, for any nonnegative translation invariant function $w(A)$ giving weight to a subset $A\subset\mathbb{Z}^d$ we have the following estimate on its decay

$$
\sum_{A:A\ni x_0} w(A)\int I\!P(d\tilde\eta)\left|U_{\mu;A}^{\text{fe,abs,inv}}(\tilde\eta)\right| \le C_1 + C_2\sum_{m=2}^{\infty} m^{2d-1}\bar w(m)\bar c(m)
\tag{2.11}
$$

where $\bar w(m) := w\left(\{z\in\mathbb{Z}^d; z\ge 0, |z|\le m\}\right)$ where $\ge$ denotes the lexicographic order. The constants C_1, C_2 are related to a-priori bounds on ΔH_x.

Under the stronger condition that we have bounds of the same form on the $\sup_{\substack{x,y:|x-y|=m\\ \eta_x,\eta_y\in\mathcal{H}_0}}$ $\sup_{\tilde\eta}|c_{x,y}(\eta_x,\eta_y,\tilde\eta)|$ the absolute convergence is not only a.s. but everywhere, and (2.11) holds for all realizations without the $I\!P$-integral (with non-random constants).

III. The infinite-volume conditional expectations

We start with a suitable representation of the infinite-volume conditional expectations of the joint measure.

We write $\xi = (\sigma,\eta)$ here and below, so that, for any set $A\subset\mathbb{Z}^d$ we have $\xi_A = (\sigma_A,\eta_A)$. Recall that r is the range of the defining potential Φ. We write $\overline{A} = \{y\in\mathbb{Z}^d, d(y,A)\le r\}$ for the r-neighborhood of a set A, and put $\partial A = \overline{A}\backslash A$.

Proposition 3.1: *Assume there is a set of realizations $\mathcal{H}^0 \subset \mathcal{H}$ of $I\!P$-measure one such that the quenched infinite-volume Gibbs measure $\mu[\eta]$ is a weak limit (2.2) of the quenched finite volume measures (2.1) for all $\eta \in \mathcal{H}^0$. Then, a version of the infinite-volume conditional expectation of the corresponding joint measure $K(d\sigma, d\eta) = I\!P(d\eta)\mu[\eta](d\sigma)$ is given by the formula*

$$K\left[\xi_\Lambda | \xi_{\boldsymbol{Z}^d \setminus \Lambda}\right] = \frac{\mu_\Lambda^{\mathrm{ann},\xi_{\partial\Lambda}}(\xi_\Lambda)}{\int \mu_\Lambda^{\mathrm{ann},\xi_{\partial\Lambda}}(d\tilde\eta_\Lambda) Q_\Lambda(\eta_\Lambda, \tilde\eta_\Lambda, \eta_{\boldsymbol{Z}^d\setminus\Lambda})} \tag{3.1}$$

Here $\mu_\Lambda^{\mathrm{ann},\xi_{\partial\Lambda}}(\xi_\Lambda)$ is the **trivial annealed local specification** *given by (2.5), which can be written in terms of the potential $U_A^{triv}(\sigma,\eta) = \Phi_A(\sigma,\eta) - 1_{A=\{x\}} \log \nu(\eta_x)$.*

Further we have put

$$Q_\Lambda(\eta^1_\Lambda, \eta^2_\Lambda, \eta_{\boldsymbol{Z}^d\setminus\Lambda}) := \mu[\eta^2_\Lambda \eta_{\boldsymbol{Z}^d\setminus\Lambda}](e^{-\Delta H_\Lambda(\eta^1_\Lambda, \eta^2_\Lambda, \eta_{\partial\Lambda})}) \tag{3.2}$$

According to our assumption on the measurability on $\mu[\eta]$, Q_Λ depends measurably on $\eta_{\boldsymbol{Z}^d\setminus\Lambda}$. We note the following properties

(i) $Q_\Lambda(\eta^1_\Lambda, \eta^2_\Lambda, \eta_{\boldsymbol{Z}^d\setminus\Lambda}) = \left[Q_\Lambda(\eta^2_\Lambda, \eta^1_\Lambda, \eta_{\boldsymbol{Z}^d\setminus\Lambda})\right]^{-1}$

(ii) For any $\Delta \supset \Lambda$ we have $Q_\Delta(\eta^1_\Lambda \eta_{\Delta\setminus\Lambda}, \eta^2_\Lambda \eta_{\Delta\setminus\Lambda}, \eta_{\boldsymbol{Z}^d\setminus\Delta}) = Q_\Lambda(\eta^1_\Lambda, \eta^2_\Lambda, \eta_{\boldsymbol{Z}^d\setminus\Lambda})$

(iii) For any η^3_Λ we have $\frac{Q_\Lambda(\eta^1_\Lambda, \eta^3_\Lambda, \eta_{Z^d\setminus\Lambda})}{Q_\Lambda(\eta^2_\Lambda, \eta^3_\Lambda, \eta_{Z^d\setminus\Lambda})} = Q_\Lambda(\eta^1_\Lambda, \eta^2_\Lambda, \eta_{\boldsymbol{Z}^d\setminus\Lambda})$

whenever $\eta \in \mathcal{H}^0$.

Remark: Note that, by our assumption on the a.s. convergence of the infinite-volume Gibbs measures, Q_Λ can be written in the form

$$Q_\Lambda(\eta^1_\Lambda, \eta^2_\Lambda, \eta_{\boldsymbol{Z}^d\setminus\Lambda}) = \lim_{\Lambda_N \uparrow \boldsymbol{Z}^d} \mu_{\Lambda_N}^{\sigma^{\mathrm{b.c.}}_{\partial\Lambda_N}}[\eta^2 \eta_{\overline{\Lambda_N}}]\left(e^{-\Delta H_\Lambda(\eta^1_\Lambda, \eta^2_\Lambda, \eta_{\partial\Lambda})}\right) = \lim_{\Lambda_N \uparrow \boldsymbol{Z}^d} \frac{Z_{\Lambda_N}^{\sigma^{\mathrm{b.c.}}_{\partial\Lambda_N}}[\eta^1_\Lambda \eta_{\overline{\Lambda_N}\setminus\Lambda}]}{Z_{\Lambda_N}^{\sigma^{\mathrm{b.c.}}_{\partial\Lambda_N}}[\eta^2_\Lambda \eta_{\overline{\Lambda_N}\setminus\Lambda}]} \tag{3.3}$$

with the quenched partition function

$$Z_\Lambda^{\sigma^{\mathrm{b.c.}}_{\partial\Lambda}}[\eta_{\overline{\Lambda}}] = \sum_{\sigma_\Lambda} e^{-\sum_{A: A\cap\Lambda\neq\emptyset} \Phi_A(\sigma_\Lambda \sigma^{\mathrm{b.c.}}_{\partial\Lambda}, \eta_{\overline{\Lambda}})} \tag{3.4}$$

whenever $\eta \in \mathcal{H}^0$. Morally, Q_Λ is thus a ratio of infinite-volume partition functions whose disorder variables differ in the volume Λ.

Remark: We note that formulas for the *finite volume conditional expectations* have appeared in [K6] [see Lemma 2.1, (2.4) therein]. They seem to look more complicated than the infinite-volume expression (3.1). In that paper we wanted to be able to deal also with the more general case in which we do not assume $I\!P$-a.s. convergence of the finite volume Gibbs measures, but only convergence of the finite volume joint measures. Then (3.1) is not available.

Proof: Properties (i),(ii),(iii) are clear from (3.3).

To get (3.1) we will show at first that, for the measure $K^{\sigma^{\text{b.c.}}_{\partial\Lambda_N}}_{\Lambda_N}(\sigma_{\Lambda_N},\eta_{\overline{\Lambda_N}}) := I\!P(\eta_{\overline{\Lambda_N}})\mu^{\sigma^{\text{b.c.}}_{\partial\Lambda_N}}_{\Lambda_N}[\eta_{\overline{\Lambda_N}}](\sigma_{\Lambda_N})$ on $\Omega_\Lambda\times\mathcal{H}_{\overline{\Lambda}}$ we have, for finite Λ,Δ,Λ_N with $\Lambda\subset\Delta$ and $\overline{\Delta}\subset\Lambda_N$, the formula

$$K^{\sigma^{\text{b.c.}}_{\partial\Lambda_N}}_{\Lambda_N}\left[\xi_\Lambda\middle|\xi_{\Delta\setminus\Lambda}\right] = \int K^{\sigma^{\text{b.c.}}_{\partial\Lambda_N}}_{\Lambda_N}\left[d\bar\sigma_{\Lambda_N\setminus\Delta}d\bar\eta_{\overline{\Lambda_N}\setminus\Delta}\middle|\xi_{\Delta\setminus\Lambda}\right]\frac{\mu_\Lambda^{\text{ann},\xi_{\partial\Lambda\cap\Delta}\bar\xi_{\partial\Lambda\setminus\Delta}}(\xi_\Lambda)}{\int\mu_\Lambda^{\text{ann},\xi_{\partial\Lambda\cap\Delta}\bar\xi_{\partial\Lambda\setminus\Delta}}(d\tilde\eta_\Lambda)\frac{Z^{\sigma^{\text{b.c.}}_{\partial\Lambda_N}}_{\Lambda_N}[\eta_\Lambda\eta_{\Delta\setminus\Lambda}\bar\eta_{\overline{\Lambda_N}\setminus\Delta}]}{Z^{\sigma^{\text{b.c.}}_{\partial\Lambda_N}}_{\Lambda_N}[\tilde\eta_\Lambda\eta_{\Delta\setminus\Lambda}\bar\eta_{\overline{\Lambda_N}\setminus\Delta}]}} \tag{3.5}$$

In particular the formula holds true for $\Lambda=\Delta$. Now, (3.5) is just a computation. Indeed, write

$$\begin{aligned}&K^{\sigma^{\text{b.c.}}_{\partial\Lambda_N}}_{\Lambda_N}\left[\xi_\Lambda\middle|\xi_{\Delta\setminus\Lambda}\right]\\ &=\int K^{\sigma^{\text{b.c.}}_{\partial\Lambda_N}}_{\Lambda_N}\left[d\bar\sigma_{\Lambda_N\setminus\Delta}d\bar\eta_{\overline{\Lambda_N}\setminus\Delta}\middle|\xi_{\Delta\setminus\Lambda}\right]K^{\sigma^{\text{b.c.}}_{\partial\Lambda_N}}_{\Lambda_N}\left[\xi_\Lambda\middle|\xi_{\Delta\setminus\Lambda}\bar\sigma_{\Lambda_N\setminus\Delta}\bar\eta_{\overline{\Lambda_N}\setminus\Delta}\right]\end{aligned} \tag{3.6}$$

and note that the term under the integral on the r.h.s. equals

$$\begin{aligned}&\frac{K^{\sigma^{\text{b.c.}}_{\partial\Lambda_N}}_{\Lambda_N}\left[\xi_\Lambda\xi_{\Delta\setminus\Lambda}\bar\sigma_{\Lambda_N\setminus\Delta}\bar\eta_{\overline{\Lambda_N}\setminus\Delta}\right]}{\sum_{\tilde\xi_\Lambda}K^{\sigma^{\text{b.c.}}_{\partial\Lambda_N}}_{\Lambda_N}\left[\tilde\xi_\Lambda\xi_{\Delta\setminus\Lambda}\bar\sigma_{\Lambda_N\setminus\Delta}\bar\eta_{\overline{\Lambda_N}\setminus\Delta}\right]}\\ &=\frac{I\!P(\eta_\Lambda)\mu^{\sigma^{\text{b.c.}}_{\partial\Lambda_N}}_{\Lambda_N}[\eta_\Lambda\eta_{\Delta\setminus\Lambda}\bar\eta_{\overline{\Lambda_N}\setminus\Delta}](\sigma_\Lambda\sigma_{\Delta\setminus\Lambda}\bar\sigma_{\Lambda_N\setminus\Delta})}{\sum_{\tilde\sigma_\Lambda,\tilde\eta_\Lambda}I\!P(\tilde\eta_\Lambda)\mu^{\sigma^{\text{b.c.}}_{\partial\Lambda_N}}_{\Lambda_N}[\tilde\eta_\Lambda\eta_{\Delta\setminus\Lambda}\bar\eta_{\overline{\Lambda_N}\setminus\Delta}](\tilde\sigma_\Lambda\sigma_{\Delta\setminus\Lambda}\bar\sigma_{\Lambda_N\setminus\Delta})}\end{aligned} \tag{3.7}$$

Spelling out the quenched local specifications in terms of the random potential Φ this can be rewritten in terms of the special annealed potential $U^{triv}_A(\sigma,\eta)=\Phi_A(\sigma,\eta)-1_{A=\{x\}}\log\nu(\eta_x)$ as

$$\frac{e^{-\sum_{A:A\cap\Lambda\neq\emptyset}U^{triv}_A(\sigma_\Lambda\sigma_{\Delta\setminus\Lambda}\bar\sigma_{\Lambda_N\setminus\Delta},\eta_\Lambda\eta_{\Delta\setminus\Lambda}\bar\eta_{\overline{\Lambda_N}\setminus\Delta})}}{\sum_{\tilde\sigma_\Lambda,\tilde\eta_\Lambda}e^{-\sum_{A:A\cap\Lambda\neq\emptyset}U^{triv}_A(\tilde\sigma_\Lambda\sigma_{\Delta\setminus\Lambda}\bar\sigma_{\Lambda_N\setminus\Delta},\tilde\eta_\Lambda\eta_{\Delta\setminus\Lambda}\bar\eta_{\overline{\Lambda_N}\setminus\Delta})}\frac{Z^{\sigma^{\text{b.c.}}_{\partial\Lambda_N}}_{\Lambda_N}[\eta_\Lambda\eta_{\Delta\setminus\Lambda}\bar\eta_{\overline{\Lambda_N}\setminus\Delta}]}{Z^{\sigma^{\text{b.c.}}_{\partial\Lambda_N}}_{\Lambda_N}[\tilde\eta_\Lambda\eta_{\Delta\setminus\Lambda}\bar\eta_{\overline{\Lambda_N}\setminus\Delta}]}} \tag{3.8}$$

Note that, due to cancellations for $\overline{\Delta}\subset\Lambda_N$, the U-sums do not depend on $\sigma^{\text{b.c.}}$. Note that, for $\overline{\Lambda}\subset\Delta$, (3.8) does not depends on $\bar\sigma_{\Lambda_N\setminus\Delta}$. In this case the outer integral in (3.5) reduces to an integration over the disorder variables. Note however that this is **not** a product integration! Finally, dividing both numerator and denominator of (3.8) by the annealed partition function $\sum_{\tilde\sigma_\Lambda,\tilde\eta_\Lambda}e^{-\sum_{A:A\cap\Lambda\neq\emptyset}U^{triv}_A(\tilde\sigma_\Lambda\sigma_{\Delta\setminus\Lambda}\bar\sigma_{\Lambda_N\setminus\Delta},\tilde\eta_\Lambda\eta_{\Delta\setminus\Lambda}\bar\eta_{\overline{\Lambda_N}\setminus\Delta})}$ we get the desired (3.5).

Next we claim that

$$K\left[\xi_\Lambda\middle|\xi_{\Delta\setminus\Lambda}\right]=\int K\left[d\bar\xi_{\mathbb{Z}^d\setminus\Delta}\middle|\xi_{\Delta\setminus\Lambda}\right]\frac{\mu_\Lambda^{\text{ann},\xi_{\partial\Lambda\cap\Delta}\bar\xi_{\partial\Lambda\setminus\Delta}}(\xi_\Lambda)}{\int\mu_\Lambda^{\text{ann},\xi_{\partial\Lambda\cap\Delta}\bar\xi_{\partial\Lambda\setminus\Delta}}(d\tilde\eta_\Lambda)Q_\Lambda(\eta_\Lambda,\tilde\eta_\Lambda,\eta_{\Delta\setminus\Lambda}\bar\eta_{\mathbb{Z}^d\setminus\Delta})} \tag{3.9}$$

To see this, write down (3.5) explicitly in terms of the quenched local specifications and (3.9) in terms of the infinite-volume Gibbs measure. Note that the dependence on those measures is completely local- therefore (3.9) follows by the assumption of $I\!P$-a.s. local convergence of the finite volume Gibbs measures. But from (3.9) we can conclude now, that what is under the integral on the r.h.s. must be the infinite-volume conditional expectation. More precisely, (3.1) follows from the following general measure-theoretic

Fact: *Assume that $\xi_{\mathbf{Z}^d}$ is a random field with distribution K, ξ_x taking values in a finite set, and $\tilde K\left[\xi_\Lambda | \xi_{\mathbf{Z}^d\setminus\Lambda}\right]$ is a Borel probability kernel that satisfies*

$$K\left[\xi_\Lambda|\xi_{\Delta\setminus\Lambda}\right] = \int K\left[d\bar\xi_{\mathbf{Z}^d\setminus\Delta}|\xi_{\Delta\setminus\Lambda}\right] \tilde K\left[\xi_\Lambda|\xi_{\Delta\setminus\Lambda}\bar\xi_{\mathbf{Z}^d\setminus\Delta}\right] \tag{3.10}$$

for all finite $\Delta\supset\Lambda$, where $K\left[d\bar\xi_{\mathbf{Z}^d\setminus\Delta}|\xi_{\Delta\setminus\Lambda}\right]$ is a version of the conditional expectation. Then $\tilde K\left[\xi_\Lambda|\xi_{\mathbf{Z}^d\setminus\Lambda}\right]$ is a version of the infinite-volume conditional expectation $K\left[\xi_\Lambda|\xi_{\mathbf{Z}^d\setminus\Lambda}\right]$.

We include a proof for the convenience of the reader:

$\tilde K\left[\xi_\Lambda|\xi_{\mathbf{Z}^d\setminus\Lambda}\right]$ is assumed to be $\sigma\left(\xi_{\mathbf{Z}^d\setminus\Lambda}\right)$-measurable. So, to verify the definition of the conditional expectation we have to show that, for all events $C\in\sigma\left(\xi_{\mathbf{Z}^d\setminus\Lambda}\right)$ and $A\in\sigma\left(\xi_{\mathbf{Z}^d}\right)$ we have that

$$\int_C \left(\int_A \tilde K\left[d\xi_\Lambda|\xi'_{\mathbf{Z}^d\setminus\Lambda}\right]\otimes\delta_{\xi'_{\mathbf{Z}^d\setminus\Lambda}}(d\xi_{\mathbf{Z}^d\setminus\Lambda})\right) K(d\xi'_{\mathbf{Z}^d\setminus\Lambda}) = K(A\cap C) \tag{3.11}$$

Writing A in the form $A=\sum_{\xi_\Lambda}\left(\{\xi_\Lambda\}\times A_{\xi_\Lambda}\right)$ where $A_{\xi_\Lambda}\in\sigma\left(\xi_{\mathbf{Z}^d\setminus\Lambda}\right)$ we see that this is equivalent to $\sum_{\xi_\Lambda}\int_C \tilde K\left[\xi_\Lambda|\xi'_{\mathbf{Z}^d\setminus\Lambda}\right] 1_{\xi'_{\mathbf{Z}^d\setminus\Lambda}\in A_{\xi_\Lambda}} K(d\xi'_{\mathbf{Z}^d\setminus\Lambda}) = \sum_{\xi_\Lambda} K(\{\xi_\Lambda\}\times(A_{\xi_\Lambda}\cap C))$. So, it suffices to show that, for any $B\in\sigma\left(\xi_{\mathbf{Z}^d\setminus\Lambda}\right)$ and any ξ_Λ, we have that

$$\int_B \tilde K\left[\xi_\Lambda|\xi'_{\mathbf{Z}^d\setminus\Lambda}\right] K(d\xi'_{\mathbf{Z}^d\setminus\Lambda}) = K(\{\xi_\Lambda\}\times B) \tag{3.12}$$

To see this, we apply the standard Dynkin-class argument to show an equality for all sets of a given σ-algebra, see e.g. [Co] Theorem 1.6.1 (which states that, for any $\cap$-stable set $\mathcal{F}$ of subsets, the smallest σ-algebra which contains $\mathcal{F}$ coincides with the smallest Dynkin-class which contains $\mathcal{F}$). First note that the system $\mathcal{D}$ of sets B in $\sigma\left(\xi_{\mathbf{Z}^d\setminus\Lambda}\right)$ for which this equality holds is a Dynkin class: That $\Omega\in\mathcal{D}$ follows from (3.10) for $\Delta=\Lambda$; furthermore $\mathcal{D}$ is stable under formation of complements and countable unions of pairwise disjoint sets, by the properties of the integral.

Thus we only need to prove (3.12) for the set of cylinder sets, since they form a $\cap$-stable generator of $\sigma\left(\xi_{\mathbf{Z}^d\setminus\Lambda}\right)$. It suffices to take sets of the form $B=\{\tilde\xi, \tilde\xi_{\Delta\setminus\Lambda}=\xi_{\Delta\setminus\Lambda}\}$. But note that in this case

$$\begin{aligned}\int_B \tilde K\left[\xi_\Lambda|\xi'_{\mathbf{Z}^d\setminus\Lambda}\right] K(d\xi'_{\mathbf{Z}^d\setminus\Lambda}) &= \int \tilde K\left[\xi_\Lambda|\xi_{\Delta\setminus\Lambda}\xi'_{\mathbf{Z}^d\setminus\Delta}\right] K(d\xi'_{\mathbf{Z}^d\setminus\Delta}|\xi_{\Delta\setminus\Lambda})K(\xi_{\Delta\setminus\Lambda})\\ &= K\left[\xi_\Lambda|\xi_{\Delta\setminus\Lambda}\right] K(\xi_{\Delta\setminus\Lambda}) = K(\{\xi_\Lambda\}\times B)\end{aligned} \tag{3.13}$$

where we have used the hypothesis in the second equality. This concludes the proof of the "fact" and concludes the proof of the proposition. $\diamondsuit$

IV. Construction of Potentials - Proof of the Theorems

Starting from the formula of Proposition 3.1 for the infinite-volume conditional expectations of the joint measure K we will prove Theorems 2.1 and 2.2 at the same time. A little later we will prove Theorem 2.4.

As a first consequence of Proposition 3.1 we separate the potential for the joint measures we are about to construct into an "annealed part" and a "free energy" part. We have

Lemma 4.1: *Suppose that $U^{\mathrm{ann}}(\xi)$ is a potential for the annealed system. Then we have that $U(\sigma,\eta) = U^{\mathrm{ann}}(\sigma,\eta) + U^{\mathrm{fe}}(\eta)$ generates the conditional expectations for the joint measure K if $U^{\mathrm{fe}}(\eta)$ is summable for $I\!P$-a.e η and, $I\!P$-a.s.,*

$$\lim_{\Delta\uparrow \mathbb{Z}^d} \sum_{A:A\subset\Delta, A\cap\Lambda\neq\emptyset} \Bigl(U_A^{fe}(\eta_\Lambda^1\eta_{\mathbb{Z}^d\setminus\Lambda}) - U_A^{fe}(\eta_\Lambda^2\eta_{\mathbb{Z}^d\setminus\Lambda})\Bigr) = \log Q_\Lambda(\eta_\Lambda^1,\eta_\Lambda^2,\eta_{\mathbb{Z}^d\setminus\Lambda}) \tag{4.1}$$

Proof: For finite $\Delta\supset\Lambda$ we write

$$\begin{aligned}
&\frac{e^{-\sum_{A:A\subset\Delta,A\cap\Lambda\neq\emptyset}U_A(\xi)}}{\sum_{\tilde\xi_\Lambda} e^{-\sum_{A:A\subset\Delta,A\cap\Lambda\neq\emptyset}U_A(\tilde\xi_\Lambda\xi_{\mathbb{Z}^d\setminus\Lambda})}}\\
&=\frac{e^{-\sum_{A:A\subset\Delta,A\cap\Lambda\neq\emptyset}U_A^{\mathrm{ann}}(\xi)}}{\sum_{\tilde\xi_\Lambda} e^{-\sum_{A:A\subset\Delta,A\cap\Lambda\neq\emptyset}U_A^{\mathrm{ann}}(\tilde\xi_\Lambda\xi_{\mathbb{Z}^d\setminus\Lambda})}e^{-\sum_{A:A\subset\Delta}\left(U_A^{fe}(\tilde\eta_\Lambda\eta_{\mathbb{Z}^d\setminus\Lambda})-U_A^{fe}(\eta)\right)}}\\
&=\frac{\mu_\Lambda^{\mathrm{ann},\xi_{\partial\Lambda}}(\xi_\Lambda)}{\int\mu_\Lambda^{\mathrm{ann},\xi_{\partial\Lambda}}(d\tilde\eta_\Lambda)e^{-\sum_{A:A\subset\Delta}\left(U_A^{fe}(\tilde\eta_\Lambda\eta_{\mathbb{Z}^d\setminus\Lambda})-U_A^{fe}(\eta)\right)}}
\end{aligned}\tag{4.2}$$

Here the first equality is just a resummation of sums and the second follows from normalizing by the annealed partition function. Now the claim follows from formula (3.1) for the infinite-volume conditional expectations of K by the limit $\Delta\uparrow\mathbb{Z}^d$. $\diamondsuit$

Thus we are completely reduced to the investigation of the Q-part. Hence we will define our potentials in terms of logarithms of Q_Λ's. This makes life much easier and formulas much more transparent than dealing with the full conditional probabilities of the joint measures themselves. The situation is especially nice here, since the Q- part depends only on the disorder variables and the marginal of the joint measures we consider on the disorder variables is just a product measure.

Proof of Theorem 2.1 and 2.2: Denote by α any product-measure on the disorder space. Later we will put either $\alpha = I\!P$ or $\alpha = \delta_{\hat\eta}$ for a fixed realization of the disorder $\hat\eta$, the first case corresponding to the proof of Theorem 2.1, the second case corresponding to the proof of Theorem 2.2. For the second case we assume that $\hat\eta$ is in the set of realizations for which the convergence (2.2) holds. From this follows: For all realizations which are finite volume perturbations of $\hat\eta$ the convergence (2.2) to an infinite-volume Gibbs measure with the corresponding local specification holds, too. (This is seen by splitting off the corresponding terms in the Hamiltonian and treating them as local observables.) So the l.h.s. of (2.8) is uniquely defined.

We define the 'relative energy'

$$\begin{aligned} E^\alpha_\Lambda(\eta_\Lambda) &:= \int \alpha(d\tilde\eta) \log Q_\Lambda(\eta_\Lambda, \tilde\eta_\Lambda, \tilde\eta_{\mathbb{Z}^d\setminus\Lambda}) \\ &= \int \alpha(d\tilde\eta) \log \mu[\tilde\eta](e^{-\Delta H_\Lambda(\eta_\Lambda, \tilde\eta_\Lambda, \tilde\eta_{\partial\Lambda})}) \end{aligned} \tag{4.3}$$

and define a potential by the inclusion-exclusion principle

$$\begin{aligned} U^{\mathrm{fe},\alpha}_A(\eta) &:= \sum_{\Lambda:\Lambda\subset A} (-1)^{|A\setminus\Lambda|} E^\alpha_\Lambda(\eta_\Lambda) \quad \text{so that} \\ E^\alpha_\Lambda(\eta_\Lambda) &= \sum_{A:A\subset\Lambda} U^{fe,\alpha}_A(\eta) \end{aligned} \tag{4.4}$$

We remark that the application of the inclusion-exclusion principle to define a formal potential is a classical thing that goes back even before [Koz]. Note that, by choosing $\alpha = \delta_{\hat\eta}$, (4.3) becomes an expectation w.r.t. a non-random system and thus, for a suitable translation-invariant realization $\hat\eta$, might even be amenable to explicit computations in certain cases. Of course, for $\alpha = I\!P$, (4.3) involves the full disorder-dependence of the random Gibbs measure and will hardly ever be suitable for explicit computations.

Note that the family of random variables E^α_Λ, indexed by finite subsets $\Lambda\subset\mathbb{Z}^d$, is a **martingale w.r.t. the product measure α.** This means that, for each $\Delta\supset\Lambda$,

$$\int \alpha(d\bar\eta) E^\alpha_\Delta(\eta_\Lambda\bar\eta_{\Delta\setminus\Lambda}) = E^\alpha_\Lambda(\eta_\Lambda), \quad E^\alpha_\emptyset := \int \alpha(d\tilde\eta) E^\alpha_\Lambda(\tilde\eta_\Lambda) = 0 \tag{4.5}$$

Indeed, we have by Proposition 3.1 (iii)

$$\begin{aligned} &\int \alpha(d\bar\eta) \int \alpha(d\tilde\eta) \log Q_\Delta(\eta_\Lambda\bar\eta_{\Delta\setminus\Lambda}, \tilde\eta_\Delta, \tilde\eta_{Z^d\setminus\Delta}) \\ &= \int \alpha(d\bar\eta) \int \alpha(d\tilde\eta) \Big(\log Q_\Delta(\eta_\Lambda\bar\eta_{\Delta\setminus\Lambda}, \bar\eta_\Delta, \tilde\eta_{Z^d\setminus\Delta}) + \log Q_\Delta(\bar\eta_\Delta, \eta'_\Delta, \tilde\eta_{Z^d\setminus\Delta}) + \log Q_\Delta(\eta'_\Delta, \tilde\eta_\Delta, \tilde\eta_{Z^d\setminus\Delta}) \Big) \end{aligned} \tag{4.6}$$

for any fixed η'. The last two terms cancel, due to Proposition 3.1 (i) and the first term equals $E^{\alpha}_{\Lambda}(\eta_\Lambda)$, due to (ii), as desired. Note that this works also in the case $\alpha = \delta_{\hat\eta}$ since we assumed weak convergence for the point $\hat\eta$!

From this it follows easily from the usual play with signed sums that, in fact, the potential $U^{\mathrm{fe},\alpha}$ is α-normalized as a potential on the disorder space, i.e. $\int \alpha_x(d\tilde\eta_x) U^{fe,\alpha}_A(\eta_{A\setminus x}\tilde\eta_x) = 0$ whenever $x \in A$.

Next, to prove that the potential converges, write

$$\begin{aligned}\sum_{A:A\subset\Delta, A\cap\Lambda\neq\emptyset} U^{fe,\alpha}_A(\eta) &= \sum_{A:A\subset\Delta} U^{fe,\alpha}_A(\eta) - \sum_{A:A\subset\Delta\setminus\Lambda} U^{\alpha,fe}_A(\eta)\\ &= E^{\alpha}_{\Delta}(\eta) - E^{\alpha}_{\Delta\setminus\Lambda}(\eta)\\ &= \int \alpha(d\tilde\eta)\log\frac{Q_\Delta(\eta_\Delta,\tilde\eta_\Delta,\tilde\eta_{\mathbb{Z}^d\setminus\Delta})}{Q_\Delta(\tilde\eta_\Lambda\eta_{\Delta\setminus\Lambda},\tilde\eta_\Delta,\tilde\eta_{\mathbb{Z}^d\setminus\Delta})}\\ &= \int \alpha(d\tilde\eta)\log Q_\Lambda(\eta_\Lambda,\tilde\eta_\Lambda,\eta_{\Delta\setminus\Lambda}\tilde\eta_{\mathbb{Z}^d\setminus\Delta})\end{aligned}\tag{4.7}$$

The second equality is (4.4) and for the next two equalities we have used properties (ii) and (iii) for Q. The important point that exploits the nature of α being a product measure is the convergence statement

$$\lim_{\Delta\uparrow\mathbb{Z}^d}\int \alpha(d\tilde\eta)\log Q_\Lambda(\eta^1_\Lambda,\eta^2_\Lambda,\eta_{\Delta\setminus\Lambda}\tilde\eta_{\mathbb{Z}^d\setminus\Delta}) = \log Q_\Lambda(\eta^1_\Lambda,\eta^2_\Lambda,\eta_{\mathbb{Z}^d\setminus\Lambda}) \quad \text{for} \quad \alpha\text{-a.e. } \eta \tag{4.8}$$

This follows by the martingale convergence theorem, since, for any fixed finite $\Lambda\subset\mathbb{Z}^d$ and fixed $\eta^1_\Lambda, \eta^2_\Lambda$ the expression under the limit on the l.h.s indexed by finite subsets $\Delta\subset\mathbb{Z}^d$ s.t. $\Delta\supset\Lambda$, is a martingale w.r.t the distribution given by α.

Proof of Theorem 2.1: We put $\alpha = \mathbb{P}$. Then we see from (4.7) and (4.8) that the potential converges with $\Delta\uparrow\mathbb{Z}^d$ for $\mathbb{P}$-a.e. η. Since $\mathbb{P}$ is the marginal of K on the disorder-space, this is exactly what we want.

Proof of Theorem 2.2: We put $\alpha = \delta_{\hat\eta}$ where $\hat\eta$ is the assumed direction of continuity. In this case the r.h.s. of (4.7) is just $Q_\Lambda(\eta_\Lambda,\hat\eta_\Lambda,\eta_{\Delta\setminus\Lambda}\hat\eta_{\mathbb{Z}^d\setminus\Delta})$. Using property (iii) for Q_Λ we may rewrite this as a telescoping sum $\sum_{x\in\Lambda} Q_\Lambda(\eta_{\Lambda_{\leq x}},\eta_{\Lambda_{<x}},\eta_{\Delta\setminus\Lambda}\hat\eta_{\mathbb{Z}^d\setminus\Delta})$. Here we have put the lexicographic order on $\mathbb{Z}^d$ and written $\Lambda_{\leq x} = \{z\in\Lambda; z\leq x\}$ (and the analogous notation for "<"). Thus we see that (2.7) really implies convergence of the potential with $\Delta\uparrow\mathbb{Z}^d$.

Next we prove that the potential generates the infinite-volume conditional expectations of the joint measure K. We must verify hypothesis (4.1) of Lemma 4.1. We have

$$\begin{aligned}
&\sum_{A:A\subset\Delta}\Big(U_A^{fe}(\eta_\Lambda^1\eta_{\mathbb{Z}^d\setminus\Lambda})-U_A^{fe}(\eta_\Lambda^2\eta_{\mathbb{Z}^d\setminus\Lambda})\Big)\\
&=E_\Delta^\alpha(\eta_\Lambda^1\eta_{\Delta\setminus\Lambda})-E_\Delta^\alpha(\eta_\Lambda^2\eta_{\Delta\setminus\Lambda})\\
&=\int\alpha(d\tilde\eta_{\mathbb{Z}^d})\log\frac{Q_\Delta(\eta_\Lambda^1\eta_{\Delta\setminus\Lambda},\tilde\eta_\Delta,\tilde\eta_{\mathbb{Z}^d\setminus\Delta})}{Q_\Delta(\eta_\Lambda^2\eta_{\Delta\setminus\Lambda},\tilde\eta_\Delta,\tilde\eta_{\mathbb{Z}^d\setminus\Delta})}\\
&=\int\alpha(d\tilde\eta_{\mathbb{Z}^d})\log Q_\Lambda(\eta_\Lambda^1,\eta_\Lambda^2,\eta_{\Delta\setminus\Lambda}\tilde\eta_{\mathbb{Z}^d\setminus\Delta})
\end{aligned}\tag{4.9}$$

But, recalling (4.8), the proof of (4.1) is the same as that of the convergence of the potential, in the respective cases of Theorem 2.1 and Theorem 2.2. This concludes the proof of Theorems 2.1 and 2.2. The convergence statement of Corollary 2 follows from (4.7) by integration over η w.r.t. $I\!P$. In fact, we see that $\sum_{A:A\cap\Lambda\neq\emptyset}\int I\!P(d\tilde\eta)V_{\mu;A}^{\text{fe}}(\tilde\eta)$ equals the finite quantity $\int I\!P(d\eta)\log Q_\Lambda(\tilde\eta_\Lambda,\hat\eta_\Lambda,\tilde\eta_{\mathbb{Z}^d\setminus\Lambda})$. Finally we also note that, assuming continuity of Q everywhere, we have even pointwise convergence of (4.8) for both choices of α. This proves the first convergence statement after (2.9). $\diamondsuit$

A general remark about resummed potentials:

The potentials used in the proofs of Theorem 2.3 and Theorem 2.4 are obtained by resumming the supports of the α-normalized potential $U_A^{\alpha,\text{fe}}(\eta)$. The general construction is the following: Denote by P the set of finite subsets of $\mathbb{Z}^d$ and let $P=\bigcup_a P_a$ be a disjoint union s.t. (i) $C_a:=\bigcup_{A:A\subset P_a}A$ is finite for every a, and (ii) there exists a net of finite sets $\Delta_\beta\subset\mathbb{Z}^d$ s.t. $\lim_\beta\Delta_\beta=\mathbb{Z}^d$ and: for all finite Λ, we have that, for sufficiently large Δ_β, for all $A\subset\Delta_\beta$ s.t. $A\cap\Lambda\neq\emptyset$ there exists an a with $C_a\subset\Delta_\beta$ s.t. $A\in P_a$. Then $U_C^{\alpha,\text{fe, gr}}(\eta)$, defined by

$$U_{C_a}^{\alpha,\text{fe, gr}}(\eta):=\sum_{A:A\subset P_a}U_C^{\alpha,\text{fe}}(\eta),\quad U_C^{\alpha,\text{gr}}(\eta):=0\text{ if }C\neq C_a\text{ for all }a\tag{4.10}$$

is called the **resummed potential** corresponding to the given decomposition of P. The reason for the complicated looking requirement (ii) is that one has

Lemma 4.2: *Suppose that $U_C^{\alpha,\text{fe, gr}}(\eta)$ is a resummed potential obtained from the α-normalized free energy potential $U_C^{\alpha,\text{fe}}(\eta)$ that converges* **absolutely** *for $I\!P$-a.e. η. Then $U(\sigma,\eta)=U^{\text{ann}}(\sigma,\eta)+U^{\alpha,\text{fe, gr}}(\eta)$ generates the conditional expectations for the joint measure K (for any annealed potential), if the α-normalized potential does.*

Proof: For any fixed Λ we have that, for any sufficiently large Δ_β,

$$\begin{aligned}
&\sum_{C:C\subset\Delta_\beta,C\cap\Lambda\neq\emptyset}\big(U_C^{\alpha,\text{fe, gr}}(\eta_\Lambda^1\eta_{\mathbb{Z}^d\setminus\Lambda})-U_C^{\alpha,\text{fe, gr}}(\eta_\Lambda^2\eta_{\mathbb{Z}^d\setminus\Lambda})\big)\\
&=\sum_{A:A\subset\Delta_\beta,A\cap\Lambda\neq\emptyset}\big(U_A^{\alpha,\text{fe}}(\eta_\Lambda^1\eta_{\mathbb{Z}^d\setminus\Lambda})-U_A^{\alpha,\text{fe}}(\eta_\Lambda^2\eta_{\mathbb{Z}^d\setminus\Lambda})\big)
\end{aligned}\tag{4.11}$$

This is clear, since, for every term in the right sum there is precisely one term in the left sum containing its contribution, due to property (ii). Conversely, those contributions on the l.h.s. coming from A's that don't intersect Λ cancel because the field configurations agree outside of Λ. Thus, the l.h.s. converges to the r.h.s. of (4.1) along the net Δ_β. By the hypothesis of absolute convergence this implies convergence for any sequence $\Delta \uparrow \infty$, which proves the claim, by Lemma 4.1.$\diamondsuit$

The resummations used in the proofs of Theorem 2.3 and 2.4 were invented already by [Koz] and used in various publications since then. They are of the following general form. Take $\leq$ any total order of the lattice points in $\mathbb{Z}^d$. Let, for any lattice point $x \in \mathbb{Z}^d$, an increasing sequence of finite subsets $A_{x,m} \subset \{y : y \geq x\}$, $m = 1, 2, \dots$ be given s.t. $\bigcup_m A_{x,m} = \{y : y \geq x\}$. Put $A_{x,m=0} = \emptyset$ and define $P_{x,m} := \{A : x \in A \subset A_{x,m}, A \cap (A_{x,m} \setminus A_{x,m-1}) \neq \emptyset\}$. The second condition for the sum is empty for $m = 1$. Then $\bigcup_{x,m} P_{x,m} = P$ is a disjoint union and condition (i) is satisfied. Indeed, to see (ii), consider the family $\Delta_{\underline{m}} = \bigcup_{x \in \mathbb{Z}^d} A_{x,m_x}$ where $\underline{m} = (m_x)_{x \in \mathbb{Z}^d}$ is an integer vector s.t. only finitely many of the m_x's are nonzero.

Proof of Theorem 2.3: By Lemma 4.2 it suffices to show a.s. summability of a certain resummed potential. The proof of this statement essentially relies on an L^1-statement corresponding to the convergence result (4.8). In order to explain why this ensures the existence of an a.s. summable potential, however, we have to write down explicit formulas. Let $x \mapsto \#(x)$ denote a one-to-one map from $\mathbb{Z}^d$ to the integers $\{1, 2, \dots\}$. (The reader may think of some spiraling order.) Then the L^1-martingale convergence theorem gives us that

$$\begin{aligned} &\int \mathbb{P}(d\eta) \Big| \int \mathbb{P}(d\tilde\eta) \log Q_x(\eta_x, \tilde\eta_x, \tilde\eta_{\{y:1 \leq \#(y) < \#(x)\}} \eta_{\{y:\#(x) < \#(y) \leq r\}} \tilde\eta_{\{y:\#(y) > r\}}) \\ &\quad - \int \mathbb{P}(d\tilde\eta) \log Q_x(\eta_x, \tilde\eta_x, \tilde\eta_{\{y:1 \leq \#(y) < \#(x)\}} \eta_{\{y:\#(y) > \#(x)\}}) \Big| =: \epsilon_x(r) \downarrow 0 \end{aligned} \tag{4.12}$$

with $r \uparrow \infty$, for any fixed x. This is clear, since the first line of the expression under the modulus is a martingale w.r.t. to the parameter r, for any fixed x and fixed η_x.

Take some subsequence $r(n)$ of the integers, to be defined below. For $x \geq 1$, $m \geq 1$ define $A_{x,m} := \{z \in \mathbb{Z}^d, \#(x) \leq \#(z) \leq r(\#(x) + m)\}$, put also $A_{x,m=0} = \emptyset$. Starting from general α, let us define the resummed potential by the formula corresponding to (4.10), i.e.

$$U^{\alpha,\text{fe, abs}}_{A_{x,m}}(\eta) := \sum_{\substack{A: x \in A \subset A_{x,m} \\ A \cap (A_{x,m} \setminus A_{x,m-1}) \neq \emptyset}} U^{\alpha,\text{fe}}_A(\eta), \quad U^{\alpha,\text{fe, abs}}_C(\eta) = 0 \text{ otherwise} \tag{4.13}$$

for all $x \in \mathbb{Z}^d$ and $m \geq 1$. Then we have for $m \geq 2$

$$\begin{aligned} &U^{\alpha,\text{fe, abs}}_{A_{x,m}}(\eta) = E^\alpha_{A_{x,m}}(\eta) - E^\alpha_{A_{x,m-1}}(\eta) - E^\alpha_{A_{x,m}\setminus x}(\eta) + E^\alpha_{A_{x,m-1}\setminus x}(\eta) \\ &= \int \alpha(d\tilde\eta_{\mathbb{Z}^d}) \log \frac{Q_{A_{x,m}}(\eta_{A_{x,m}}, \tilde\eta_{A_{x,m}}, \tilde\eta_{\mathbb{Z}^d \setminus A_{x,m}}) Q_{A_{x,m}}(\eta_{A_{x,m-1}\setminus x} \tilde\eta_{A_{x,m} \setminus (A_{x,m-1}\setminus x)}, \tilde\eta_{A_{x,m}}, \tilde\eta_{\mathbb{Z}^d \setminus A_{x,m}})}{Q_{A_{x,m}}(\eta_{A_{x,m-1}} \tilde\eta_{A_{x,m}\setminus A_{x,m-1}}, \tilde\eta_{A_{x,m}}, \tilde\eta_{\mathbb{Z}^d \setminus A_{x,m}}) Q_{A_{x,m}}(\eta_{A_{x,m}\setminus x} \tilde\eta_x, \tilde\eta_{A_{x,m}}, \tilde\eta_{\mathbb{Z}^d \setminus A_{x,m}})} \end{aligned} \tag{4.14}$$

In the first line we have used the expression of the relative energies in terms of the potential. In the last line we have used the definition of the relative energies and property (iii) for Q. Again, by (iii), this can be rewritten as

$$U^{\alpha,\text{fe, abs}}_{A_{x,m}}(\eta)=\int\alpha(d\tilde\eta)\log\frac{Q_x(\eta_x,\tilde\eta_x,\eta_{A_{x,m}\setminus x}\tilde\eta_{\mathbf{Z}^d\setminus A_{x,m}})}{Q_x(\eta_x,\tilde\eta_x,\eta_{A_{x,m-1}\setminus x}\tilde\eta_{\mathbf{Z}^d\setminus A_{x,m-1}})}\tag{4.15}$$

The previous formula was true for any resummed potential starting from the α-normalized free energy potential. Let us switch to $\alpha = I\!P$ and drop the subscript α. Now we have from the convergence property (4.12) our main estimate:

$$\int I\!P(d\tilde\eta)\left|U^{\text{fe, abs}}_{A_{x,m}}(\tilde\eta)\right|\leq 2\epsilon_x(r(\#(x)+m-1))\tag{4.16}$$

Similar to (4.14), (4.15) we have for $m=1$

$$\begin{aligned}U^{\alpha,\text{fe, abs}}_{A_{x,1}}(\eta)&=E^{\alpha}_{A_{x,1}}(\eta)-E^{\alpha}_{A_{x,1}\setminus x}(\eta)\\&=\int\alpha(d\tilde\eta)\log\frac{Q_{A_{x,1}}(\eta_{A_{x,1}},\tilde\eta_{A_{x,1}},\tilde\eta_{\mathbf{Z}^d\setminus A_{x,1}})}{Q_{A_{x,1}}(\eta_{A_{x,1}\setminus x}\tilde\eta_x,\tilde\eta_{A_{x,1}},\tilde\eta_{\mathbf{Z}^d\setminus A_{x,1}})}\\&=\int\alpha(d\tilde\eta)\log Q_x(\eta_x,\tilde\eta_x,\eta_{A_{x,1}\setminus x}\tilde\eta_{\mathbf{Z}^d\setminus A_{x,1}})\end{aligned}\tag{4.17}$$

This is uniformly bounded in modulus by some constant $Const_1$. From the last two estimates one concludes that

$$\begin{aligned}&\sum_{C:C\ni x}\int I\!P(d\tilde\eta)\,|U^{\text{fe, abs}}_{C}(\tilde\eta)|\leq\sum_{y:\#(y)\leq\#(x)}\sum_{m=1}^{\infty}\int I\!P(d\tilde\eta)\left|U^{\text{fe, abs}}_{A_{x,m}}(\tilde\eta)\right|\\&\leq Const_1|\{y,\#(y)\leq\#(x)\}|+2\sum_{y:\#(y)\leq\#(x)}\sum_{m=2}^{\infty}\epsilon_y(r(\#(y)+m-1))\end{aligned}\tag{4.18}$$

But, it is a simple matter to convince oneself that it is possible to choose a subsequence $r(m)$ of the integers s.t. the m-sum is finite for all y. (In fact, from $\epsilon_x(r)\downarrow 0$ one can find a subsequence $r(n)$ s.t. even $\sum_{n=1}^{\infty}\epsilon_y(r(n))<\infty$ for all y.) This completes the definition of the potential and proves $I\!P$-integrability and thus, in particular, $I\!P$-a.s. summability.$\diamondsuit$

The readers may check for themselves that one may rerun the proof for both choices of α under the hypothesis of continuity of Q everywhere. This proves the strengthened version of Theorem 2.3 promised after (2.9). One may however **not** rerun the proof for $\alpha=\delta_{\hat\eta}$ without further assumptions other than the continuity of Q_x in the direction $\hat\eta$ with the hope to obtain an absolutely summable potential. This is because the speed of convergence of the analogue of (4.12) (obtained by replacing $I\!P$ by $\delta_{\hat\eta}$) may be nonuniform in η in this case.

Proof of Theorem 2.4:

This time, denote $A_{x,m} := \{z \in \mathbb{Z}^d; z \geq x, |z-x| \leq m\}$ and define the potential by the same formula (4.13), with the new A's. Then (4.15) and (4.17) stay true. (4.17) is uniformly bounded. The potential can be rewritten in terms of correlations. Introduce $Q_{x,m,\leq y} := L_{x,m-1} \cup \{z \in L_{x,m} \backslash L_{x,m-1}; z \leq y\}$. Then, for $m \geq 2$ we have

$$U^{\text{fe,abs,inv}}_{L_{x,m}}(\eta) = \sum_{y \in L_{x,m} \backslash L_{x,m-1}} \Bigl(E^\alpha(\eta^{Q_{x,m,\leq y}}) - E^\alpha(\eta^{Q_{x,m,<y}}) - E^\alpha(\eta^{Q_{x,m,\leq y} \backslash x}) + E^\alpha(\eta^{Q_{x,m,<y} \backslash x}) \Bigr) \tag{4.19}$$

The term in brackets can be expressed as

$$-\int \alpha(d\tilde\eta) \log \frac{\mu[\eta^{Q_{x,m,<y}\backslash x}]\left(e^{-\Delta H_{\{x,y\}}(\eta_{\{x,y\}},\tilde\eta_{\{x,y\}},\eta^{Q_{x,m,<y}\backslash x}\big|_{\partial\{x,y\}})}\right)}{\mu[\eta^{Q_{x,m,<y}\backslash x}]\left(e^{-\Delta H_x(\eta_x,\tilde\eta_x,\eta^{Q_{x,m,<y}\backslash x}\big|_{\partial x})}\right) \mu[\eta^{Q_{x,m,<y}\backslash x}]\left(e^{-\Delta H_y(\eta_y,\tilde\eta_y,\eta^{Q_{x,m,<y}\backslash x}\big|_{\partial y})}\right)} \tag{4.20}$$

where we have used the notation $\eta^A := (\eta_A \tilde\eta_{\mathbb{Z}^d \backslash A})$. Note that this gives a $\tilde\eta$-dependence for the α-integral. So we get that η-expectation of the modulus of the l.h.s. is bounded from above by

$$\begin{aligned}
&\int \alpha(d\eta) \left| E^\alpha(\eta^{Q_{x,m,\leq y}}) - E^\alpha(\eta^{Q_{x,m,<y}}) - E^\alpha(\eta^{Q_{x,m,\leq y}\backslash x}) + E^\alpha(\eta^{Q_{x,m,<y}\backslash x}) \right| \\
&\leq Const \int \alpha(d\eta) \Bigg| \int \alpha(d\tilde\eta) \mu[\eta^{Q_{x,m,<y}\backslash x}]\left(e^{-\Delta H_{\{x,y\}}(\eta_{\{x,y\}},\tilde\eta_{\{x,y\}},\eta^{Q_{x,m,<y}\backslash x}\big|_{\partial\{x,y\}})}\right) \\
&\quad - \int \alpha(d\tilde\eta) \mu[\eta^{Q_{x,m,<y}\backslash x}]\left(e^{-\Delta H_x(\eta_x,\tilde\eta_x,\eta^{Q_{x,m,<y}\backslash x}\big|_{\partial x})}\right) \mu[\eta^{Q_{x,m,<y}\backslash x}]\left(e^{-\Delta H_y(\eta_y,\tilde\eta_y,\eta^{Q_{x,m,<y}\backslash x}\big|_{\partial y})}\right) \Bigg|
\end{aligned} \tag{4.21}$$

where, as always, we have used that ΔH_x is uniformly bounded to drop the logarithm. Let us now switch to the case $\alpha = I\!P$. We use the inequality $|\int f| \leq \int |f|$ for the $\tilde\eta$-integration to see that the r.h.s. is bounded from above by $Const \int I\!P(d\tilde\eta)\, |c_{x,y}(\eta_x, \eta_y, \tilde\eta)|$, the latter quantity being defined in (2.10). Recalling $\bar c(m) := \sup_{\substack{x,y: |x-y|=m \\ \eta_x,\eta_y \in \mathcal{H}_0}} \int I\!P(d\tilde\eta)\, |c_{x,y}(\eta_x, \eta_y, \tilde\eta)|$ we have from this and (4.21) that

$$\int I\!P(d\eta) \left| U^{\text{fe,abs,inv}}_{L_{x,m}}(\eta) \right| \leq Const\, |L_{x,m} \backslash L_{x,m-1}|\, \bar c(m) \leq Const' m^{d-1} \bar c(m) \tag{4.22}$$

But this gives

$$\begin{aligned}
&\sum_{A: A \ni x_0} w(A) \int I\!P(d\tilde\eta)\, |U^{\text{fe,abs,inv}}_A(\tilde\eta)| \\
&\leq \sum_{m=1}^\infty \sum_{y: |y-x_0| \leq m} w(A_{y,m}) \int I\!P(d\tilde\eta) \left| U^{\text{fe,abs,inv}}_{A_{y,m}}(\tilde\eta) \right| \leq Const_1 + Const_2 \sum_{m=2}^\infty m^{2d-1} w(A_{0,m}) \bar c(m)
\end{aligned} \tag{4.23}$$

which finishes the proof. $\diamondsuit$

We remark that the trick to relate some formal potential to expectations of certain observables by a telescoping [as in (4.19), (4.20)] was used in various papers before. Observe e.g. the analogy to the recent [MRSM] where a.s. strongly decaying potentials for renormalized measures of low temperature spin systems were constructed.

V. Examples

The results of Theorems 2.1 and 2.3 are general existence results that always apply. Let us however also see what the more specific assumptions needed for the convergence of the vacuum potential and the strengthenings of Theorems 2.1,2.3 given after (2.9) and in Theorem 2.4 mean in the examples of the (i) random field Ising model, (ii) Ising models with random couplings, and the (iii) diluted Ising ferromagnet. These examples were discussed already in [K6] w.r.t the question of almost Gibbsianness.

(i) The Random-Field Ising Model: The single spin space for the variables σ_x is $\Omega_0 = \{-1, 1\}$. The disorder variables are given by the random fields η_x that are i.i.d. with single-site distribution ν that is supported on a finite set $\mathcal{H}_0$ and assumed to be symmetric. The defining potential $\Phi(\sigma, \eta)$ is given by $\Phi_{\{x,y\}}(\sigma, \eta) = -J\sigma_x\sigma_y$ for nearest neighbors $x, y \in \mathbb{Z}^d$, $\Phi_{\{x\}}(\sigma, \eta) = -h\eta_x\sigma_x$, and $\Phi_A = 0$ else. Note that $e^{-\Delta H_x(\sigma_x,\eta_x^1,\eta_x^2)} = e^{h(\eta_x^1-\eta_x^2)\sigma_x} = e^{h(\eta_x^2-\eta_x^1)} + 2\sinh h(\eta_x^1 - \eta_x^2)\, 1_{\sigma_x=1}$. Then, treating this exponential as an observable and using the 'finite volume perturbation formula' as in [K6] we see the following. Condition (2.8) (giving the convergence of the vacuum potential) holds if and only if

$$\lim_{\Lambda\uparrow\mathbb{Z}^d} \mu[\eta_\Lambda\hat\eta_{\mathbb{Z}^d\setminus\Lambda}](\tilde\sigma_x = 1) = \mu[\eta](\tilde\sigma_x = 1) \tag{5.1}$$

for η_x, for all x, for $\mathbb{P}$-a.e. η. (Here, as always, we used the notation that spins that are integrated are decorated with tildes.) This is true for any measurable infinite-volume Gibbs measure $\mu[\eta]$ which is obtained as a weak limit with a non-random boundary condition. We note that whether (5.1) holds is independent of η_x. Similarly, condition (2.9) (giving continuity of the conditional expectations) holds, whenever

$$\lim_{\Lambda\uparrow\mathbb{Z}^d} \sup_{\hat\eta} \left|\mu[\eta_\Lambda\hat\eta_{\mathbb{Z}^d\setminus\Lambda}](\tilde\sigma_x = 1) - \mu[\eta](\tilde\sigma_x = 1)\right| = 0 \tag{5.2}$$

From this we have

Corollary to Theorem 2.2: *For any choice of the parameters of the model, the joint measure corresponding to the ferromagnetic plus-state has an a.s. convergent vacuum potential with* *vacuum* (η^+,σ). *Here* η^+ *is the configuration taking the maximum of the possible values of the magnetic field for all sites* x *and* σ *is an arbitrary spin-configuration.*

Corollary to Theorems 2.1,2.3: *Suppose that* $\lim_{\Lambda\uparrow\boldsymbol{Z}^d}\mu^+_\Lambda[\eta_\Lambda](\tilde\sigma_x=1)=\lim_{\Lambda\uparrow\boldsymbol{Z}^d}\mu^-_\Lambda[\eta_\Lambda](\tilde\sigma_x=1)$ *for all choices of the magnetic fields* $\eta\in\mathcal{H}$. *Here the expressions under the limit refer to the finite volume Gibbs-measures with* $+$ *(resp.* $-$*) boundary condition.*

Then the corresponding (unique) joint measure is Gibbs and the potentials of Theorems 2.1 and 2.2 are both convergent everywhere. There is also a potential of the form announced in Theorem 2.3 that is absolutely convergent everywhere.

Proof of Corollaries: It is known that the limit $\mu^+[\eta]=\lim_{\Lambda\uparrow\boldsymbol{Z}^d}\mu^+_\Lambda[\eta_\Lambda]$ exists for any choice of the parameters and any configuration of the quenched random fields η_x, due to monotonicity reasons. To prove the first Corollary we show that (5.1) holds for μ^+ and $\hat\eta=\eta^+$ and **any** η. To see this use the fact that the function $(\eta,\sigma^{\mathrm{bc}})\mapsto\mu^{\sigma^{\mathrm{bc}}}_\Lambda[\eta_\Lambda](\tilde\sigma_x=1)$ is monotone (w.r.t. the partial order of its arguments obtained by site-wise comparison.) From this we have

$$\mu^+[\eta](\tilde\sigma_x=1)=\limsup_{\Lambda\uparrow\boldsymbol{Z}^d}\mu^+_\Lambda[\eta_\Lambda](\tilde\sigma_x=1)\geq\limsup_{\Lambda\uparrow\boldsymbol{Z}^d}\mu^+[\eta_\Lambda\eta^+_{\boldsymbol{Z}^d\setminus\Lambda}](\tilde\sigma_x=1)\tag{5.3}$$

for any η where inequality under the limsup follows from the DLR-equation and the monotonicity. Additionally we have the converse estimate that follows from

$$\mu^+[\eta](\tilde\sigma_x=1)=\lim_{\Lambda_2\uparrow\boldsymbol{Z}^d}\mu^+_{\Lambda_2}[\eta_{\Lambda_2}](\tilde\sigma_x=1)\leq\lim_{\Lambda_2\uparrow\boldsymbol{Z}^d}\mu^+_{\Lambda_2}[\eta_\Lambda\eta^+_{\Lambda_2\setminus\Lambda}](\tilde\sigma_x=1)=\mu^+[\eta_\Lambda\eta^+_{\boldsymbol{Z}^d\setminus\Lambda}](\tilde\sigma_x=1)\tag{5.4}$$

by taking the lim inf over Λ. This proves the claim. The other Corollary follows from the remark after (2.9) and the fact that (5.2) follows from the hypothesis by $\mu^-_\Lambda[\eta_\Lambda](\tilde\sigma_x=1)\leq\mu[\eta_\Lambda\hat\eta_{\boldsymbol{Z}^d\setminus\Lambda}](\tilde\sigma_x=1)\leq\mu^+_\Lambda[\eta_\Lambda](\tilde\sigma_x=1)$ for any $\hat\eta$.$\diamondsuit$

Next we discuss the hypothesis of Theorem 2.4 giving decay of a translation invariant potential. Again, using the special form of the single-site perturbation of the Hamiltonian, it is not difficult to see that we have

$$\bar c(m)\leq Const\sup_{x,y:|x-y|=m}\int I\!P(d\tilde\eta)\,|\mu[\tilde\eta](\tilde\sigma_x\tilde\sigma_y)-\mu[\tilde\eta](\tilde\sigma_x)\mu[\tilde\eta](\tilde\sigma_y)|\tag{5.5}$$

for $m\geq 1$. (Here the sup over the possible different choices of η_x and η_y was absorbed in the constant. To see this we used the 'finite volume perturbation formula' as in [K6] Chapter III.1)

Now, let us assume that we are in the interesting region of the parameter space where existence of ferromagnetic order is proved. I.e, let us assume that we are in dimensions $d\geq 3$ and we have small disorder and large temperature, i.e. $J>0$ sufficiently large and h/J is sufficiently small. Then, a refined analysis of the renormalization group proof of Bricmont and Kupiainen should lead to the fact that (5.5) decays faster than any power with $m \uparrow \infty$ for the plus-state $\mu^+[\eta]$. [Unfortunately this does not follow directly from the (related) statement (2.6) given under [BK] Theorem (2.1) which asserts that the quenched correlation under the $I\!P$-integral decays like $Const\,(\tilde\eta)e^{-const\,|x-y|}$, since $Const\,(\tilde\eta)$ is unbounded.] This has to be contrasted with the fact that in this region the system was already proved to be **not** almost Gibbsian in [K6]. (The set of "bad configurations" of η even has full measure. The reason for this is that the magnetization $\mu^+[\eta](\tilde\sigma_x)$ can be made to jump for typical η by varying the signs of the field η in a large annulus arbitrarily far away from x. So, (5.2) does certainly not hold.)

In the opposite "high temperature" case where the coupling J is sufficiently small, one gets exponential decay $\bar c(m) \leq Const\, e^{-const\,|x-y|}$. In fact, stronger than that, one has an exponential bound on the random correlations in (5.5), uniformly in all realizations of the field. For small J this can be seen by a standard expansion of the nonrandom interaction term $e^{J1_{\sigma_x=\sigma_y}} = e^{J1_{\sigma_x=\sigma_y}} - 1 + 1$. Indeed, summation over the spins w.r.t. the independent measures $\nu(d\sigma_x)e^{h\eta_x\sigma_x}$ then produces an η-dependent polymer model that has exponential decay of correlations, uniformly in η. Of course, exponential decay of quenched correlations, uniformly in the realization of the fields, always holds in one dimension. This can be seen (e.g.) by disagreement percolation arguments. By the remark after Theorem 2.4 this implies that the joint measure is Gibbsian with an interaction potential that is superpolynomially decaying everywhere.

(ii) Ising Models with Random Nearest Neighbor Couplings: Random Bond, EA-Spinglass: The single spin space is again $\Omega_0 = \{-1, 1\}$. Denote by $\mathcal{E} := \{(1,0,0,\dots,0), (0,1,0,\dots,0),\dots,(0,0,\dots,1)\}$ the set of nearest neighbor vectors pointing in 'positive directions'. The disorder variables (random couplings) $J_{x,e}$ take finitely many values, independently over the 'bonds' x, e. We put $\eta_x = (J_{x,e})_{e\in\mathcal{E}}$. The **joint spin** at the site x is then $\xi_x = (\sigma_x, \eta_x) = (\sigma_x, (J_{x,e})_{e\in\mathcal{E}})$. The defining potential $\Phi(\sigma,\eta)$ is given by $\Phi_{\{x,y\}}(\sigma,\eta) = -J_{x,e}\sigma_x\sigma_y$ if $y = x + e$ for some $e \in \mathcal{E}$, and $\Phi_A = 0$ else. Specific distributions of interest are a) $J_{x,e}$ takes values strictly bigger than zero (random bond ferromagnet); b) $J_{x,e}$ is symmetrically distributed (EA-spinglass).

Now, the crucial observable is the correlation between nearest neighbors. We use the special form of the single site perturbation of the Hamiltonian w.r.t. η_x and similar arguments as for the random field Ising model (see [K6] chapter III.3). In this way we see that: (2.8) holds if

$$\lim_{\Lambda^*\uparrow(\mathbb{Z}^d)^*} \mu_\infty[J_{\Lambda^*}\hat J_{(\mathbb{Z}^d)^*\setminus\Lambda^*}](\tilde\sigma_x\tilde\sigma_y) = \mu_\infty[J_{(\mathbb{Z}^d)^*}](\tilde\sigma_x\tilde\sigma_y) \tag{5.6}$$

for any nearest neighbor pair x, y. Here we have written $(\mathbb{Z}^d)^*$ for the lattice of bonds of $\mathbb{Z}^d$.

Also, the condition (2.9) giving continuity of the conditional expectation holds if

$$\lim_{\Lambda^*\uparrow(\mathbb{Z}^d)^*}\sup_{\hat{J}}\left|\mu_\infty[J_{\Lambda^*}\hat{J}_{(\mathbb{Z}^d)^*\setminus\Lambda^*}](\tilde{\sigma}_x\tilde{\sigma}_y)-\mu_\infty[J_{(\mathbb{Z}^d)^*}](\tilde{\sigma}_x\tilde{\sigma}_y)\right|=0 \tag{5.7}$$

for nearest neighbors. Finally, the quantity crucial for the decay of the potential is

$$\begin{aligned}&\bar{c}(m)\\&\le Const\sup_{\substack{x,y:|x-y|=m\\e,e'\in\mathcal{E}}}\int I\!P(dJ)\left|\mu[J](\tilde{\sigma}_x\tilde{\sigma}_{x+e}\tilde{\sigma}_y\tilde{\sigma}_{y+e'})-\mu[J](\tilde{\sigma}_x\tilde{\sigma}_{x+e})\mu[J](\tilde{\sigma}_y\tilde{\sigma}_{y+e'})\right|\end{aligned} \tag{5.8}$$

for m big enough s.t. $\{x,x+e\}\cap\{y,y+e'\}$ is always empty. (Again the sup over the possible different choices of η_x and η_y was absorbed in the constant.) This quantity could be called the quenched average of the 'energy-energy'- correlation function.

We expect this to decay faster than any power in a very general situation. Exponential decay of the quantity under the modulus, uniformly in J, holds of course in a high-temperature regime where the maximum of the possible values of $|J_{x,e}|$ is sufficiently small. If this value is small enough, this can be seen by a usual high-temperature cluster expansion. This results in the existence of a translation invariant potential, whose sup-norm decays according to the remark after Theorem 2.4.

In [K6] we gave a heuristic discussion of the example of a joint measure corresponding to a random Dobrushin state for a random ferromagnet describing a stable interface between the plus and the minus state. Such states are believed to exist in $d\geq 4$ for low temperature, and weak disorder, though this is only proved in the solid-on-solid approximation (see [BoK1]). We argued that the corresponding joint measure should **not** be almost Gibbsian, if the set of possible values of the couplings contains a value that is small enough such that the corresponding homogeneous system is in the high temperature phase. Indeed, choosing this coupling in a large annulus one can decouple the inside of the system from the outside. So, the inside of the system should be in a mixture of the ferromagnetic plus resp. minus state rather than the Dobrushin state, a difference that can be observed on the nearest neighbor correlations. Nevertheless, we expect fast decay of the averaged correlations (5.8).

So, as for the random field Ising model in the phase transition regime, we should have another example of a joint measure that is not almost Gibbsian, but has an interaction potential that decays faster than any power outside of a set of measure zero, and is translation-invariant w.r.t. the shifts that preserve the joint measure (i.e. the horizontal shifts in the direction of the interface).

This potential should **not** be translation-invariant w.r.t. to shifts orthogonal to the interface. In fact, the almost surely convergent $I\!P$-normalized potential for the Dobrushin state should **not** be translation-invariant. To see this look at the explicit formula for the single site-contribution to the potential at the site x obtained by (4.3) and (4.4). It can be rewritten in terms of an integral w.r.t. the couplings $\tilde{J}$ of some function of the $\mu[\tilde{J}](\tilde{\sigma}_x\tilde{\sigma}_{x+e})$'s and local terms. Here $\mu[\tilde{J}]$ is the Dobrushin-state. Hence, the resulting expression depends explicitly on x.

This following example appears in the physical literature [Ku1,2], [MKu] and was first rigorously discussed by [EMMS] below the percolation threshold. We are a little more explicit in the discussion than in our previous examples.

(iii) The diluted random ferromagnet ('GriSing field'): The single spin space for the variables σ_x is again $\Omega_0 = \{-1, 1\}$. The disorder variables are given by the occupation numbers η_x taking values in $\{0, 1\}$, independently w.r.t. x with density $I\!P[\eta_x = 1] = p$. The disordered potential $\Phi(\sigma, \eta)$ is given by $\Phi_{\{x,y\}}(\sigma, \eta) = -J\eta_x\sigma_x\eta_y\sigma_y$ for nearest neighbors $x, y \in \mathbb{Z}^d$ and $\Phi_A = 0$ else. So the one-site variation of the Hamiltonian is $\Delta H_x(\sigma_{\overline{x}}, \eta_x^1, \eta_x^2, \eta_{\partial x}) = -J(\eta_x^1 - \eta_x^2)\sigma_x \sum_{y:d(y,x)=1} \eta_y\sigma_y$.

By the results of [EMSS] and [K6] we know that, for **any** p, for sufficiently large J, **any** weak limit of the joint measures of the GriSing random field is non-Gibbs. [EMSS] noted that, for p below p_c, the percolation threshold for ordinary site percolation, one easily obtains a potential for the joint measure by putting $U_A(\eta) = \log Z^0_{A\backslash\partial(A^c)}$ for the free energy potential if $A\backslash\partial(A^c)$ is a connected component of $\{x, \eta_x = 1\}$ and $U_A(\eta) = 0$ else. (Here Z^0_B is the partition function of the ordinary fully occupied Ising model on the set B with open boundary conditions on ∂B.) It is well-defined on the full-measure set of configurations where there is no infinite cluster and (trivially) absolutely summable on this set.

On the other hand, by the general result Theorem 2.1, we know that there is a $I\!P$-normalized potential which is convergent for $I\!P$-a.e. η for any value of p, $0 < p < 1$. By Theorem 2.3 we know that there is a (suitably regrouped) potential constructed from this potential that converges even absolutely for $I\!P$-a.e η. To be a little more specific: It is easy to see that in this case a $I\!P$-normalized potential on the disorder space can be written in the form $U^{\text{fe}}_{\mu;A}(\eta) = c_A(J,p)\prod_{x\in A}(\eta_x - p)$. From the proof of Theorem 2.1 we see that, for a given measurable Gibbs measure $\mu[\eta]$, the parameters $c_A(J,p)$ of the corresponding free energy potential are to be determined from the equations (4.3) and (4.4). Convergence of $\sum_{A:A\ni x} c_A(J,p)\prod_{y\in A}(\eta_y - p)$ for $I\!P$-a.e. η is guaranteed by Theorem 2.1. Note, on the other hand, that we certainly have that $\sum_{A:A\ni x}|c_A(J,p)|(1-p)^{|A|} = \infty$ for $p \le \frac{1}{2}$ and $\sum_{A:A\ni x}|c_A(J,p)|p^{|A|} = \infty$ for $p \ge \frac{1}{2}$ for J sufficiently large. This is clear because the above sums are just the sums over the sup-norms of the interactions and otherwise the potentials would be absolutely uniformly summable.

It is however also interesting to discuss the vacuum potentials and check the hypothesis of Theorem 2.2. We start with the potential corresponding to the 'empty' vacuum $\hat\eta^{(0)}_x \equiv 0$. It has the form $V^{\text{fe}}_{\mu;A}(\eta) = c^{(0)}_A(J)\prod_{y\in A}\eta_y$ (corresponding to [Ku2(31)]). Note that the definition of the constants $c^{(0)}_A(J)$ by (4.3) and (4.4) involves only expectations w.r.t. $\mu[\hat\eta^{(0)}]$ which is just an infinite product over symmetric Bernoulli measures. Trivially, the weak convergence (2.2) holds, and is independent of the boundary condition. So, the constants are explicitly computable up to any desired magnitude of $|A|$. In particular, they do not depend on p. Corollary 2 states that, under the hypothesis of Theorem 2.2, also the potential of the form $c^{(0)}_A(J)\left(\prod_{y\in A}\eta_y - p^{|A|}\right)$ (which corresponds to [Ku2(32)]) is an a.s. convergent potential for the joint system. The vacuum potential with 'occupied' vacuum $\hat\eta^{(1)}_x \equiv 1$ has the form $V^{\text{fe}}_{\mu;A}(\eta) = c^{(1)}_A(J)\prod_{y\in A}(\eta_y - 1)$. By (4.3), (4.4) the constants are expressed in terms of averages w.r.t. $\mu[\hat\eta^{(1)}]$ (obtained as weak limit with suitably chosen boundary condition.) We note that these constants must be such that $\sum_{A:A\ni x}|c^{(0)}_A(J)| = \infty$ and $\sum_{A:A\ni x}|c^{(1)}_A(J)| = \infty$, because $\mu[\eta]$ would be a Gibbs-measure else, as above.

$p < p_c$ (easy case): There is a unique quenched Gibbs measure $I\!\!P$-a.s. which is just the independent product over the connected components of the occupied sites (which are all finite, $I\!\!P$-a.s.) . Assuming that η is such that all connected components of occupied sites are finite, one has (2.8) for any $\hat\eta$. From this follows that the vacuum free energy potential converges, for any vacuum $\hat\eta$. In particular one has, for the empty (resp. the full) vacuum that $\sum_{A:x\in A\subset\{y\in \mathbb{Z}^d,\eta_y=1\}} c^{(0)}_A(J)$ (resp. $\sum_{A:x\in A\subset\{y\in \mathbb{Z}^d,\eta_y=0\}}(-1)^{|A|}c^{(1)}_A(J)$) converge. For the vacuum potential $V^{(0)}_A$ with empty vacuum the situation is particularly simple: We see by (4.3) and (4.4) that $V^{(0)}_A(\eta) = 0$ unless A is a subset of a connected component of $\{x \in \mathbb{Z}^d, \eta_x = 1\}$. [Because: (4.3) decomposes into a sum over the connected components of the occupied sites in Λ, i.e. $E^{(0)}_\Lambda(\eta) = \sum_i \log Z^0_{B_{\Lambda,i}(\eta)} + C_\Lambda$ where $B_{\Lambda,i}(\eta)$ are the connected components of $\{x \in \Lambda, \eta_x = 1\}$ and C_Λ does not depend on η]. This implies that $c^{(0)}_A = 0$ unless A is connected. So, $V^{(0)}_A(\eta)$ is just obtained by the decomposition of the individual logs of partition functions over all subsets A of those connected components of occupied sites and is thus a 'refinement' of the potential given just by the logs. Consequently $\sum_{A:A\ni x} V^{(0)}_A(\eta)$ contains only finitely many terms for all η such that cluster of occupied sites containing x is finite.

$p > p_c$: There is an infinite cluster of occupied sites with probability one. One may have different Gibbs measures on this infinite cluster, including the ferromagnetic ones, and also, in sufficiently high dimensions, for p sufficiently close to one, and low temperature, Dobrushin type interface states (the latter is only partially proved [BoK1]).

Let us assume at first that p, J are such that we have a ferromagnetic plus state $\mu^+[\eta]$ for $I\!\!P$-a.e. η. We look at the vacuum potential with empty vacuum, given by the same p-

independent formulas as for the $p < p_c$ case in terms of coupling constants $c_A^{(0)}$ for connected subsets $A \subset \mathbb{Z}^d$. Next we assume that η is such that the finite volume Gibbs-measures with open boundary conditions converge to the symmetric mixture $\frac{1}{2}\left(\mu^+[\eta] + \mu^-[\eta]\right)$. But, this means that $\mu[\eta_\Lambda \hat{\eta}^0_{\mathbb{Z}^d \setminus \Lambda}] \to \frac{1}{2}\left(\mu^+[\eta] + \mu^-[\eta]\right)$, because, on Λ, the l.h.s. is nothing but the finite volume Gibbs measure with open boundary conditions on $\Lambda \cap \{x \in \mathbb{Z}^d, \eta_x = 1\}$. Thus, the r.h.s. differs from the plus state as a measure, so there is **no** continuity on the level of measures. However, since the observable conjugate to the disorder variables is symmetric in σ, the corresponding expectations are the same for the plus and the minus state and we have (2.8), i.e. continuity on the level of the Q's. Assuming that the set of η's with the above property is full measure, the vacuum potential converges $\mathbb{P}$-a.s. and the corresponding joint potential describes the joint measure corresponding to the ferromagnetic plus state (and also the minus state). Conversely we have

Proposition 5.1: *Consider the dilute Ising ferrogmanet, at any fixed $J > 0$. Assume that there is a convergent free energy vacuum potential with empty vacuum $\hat{\eta}_x = 0$ for all x for the joint measure corresponding to a given Gibbs-measure $\mu[\eta]$ of the form*

$$U_A^{\mathrm{fe},0}(\eta) := c_A^{(0)} \prod_{x \in A} \eta_x \tag{5.9}$$

where A is running over the connected subsets of $\mathbb{Z}^d$. Then we must have

$$c_A^{(0)} = \sum_{\Lambda : \Lambda \subset A} (-1)^{|A \setminus \Lambda|} \log \frac{Z_\Lambda^0}{2^{|\Lambda|}} \tag{5.10}$$

where, as above, Z_Λ^0 is the partition function of the fully occupied model in Λ with zero boundary conditions. In particular, if two (possibly different) Gibbs-measures corresponding to the same J both have a potential of the form (5.9), it must be the same.

The proof is given below. Applying the proposition to the random Dobrushin (interface) state we see that we expect a different scenario for the corresponding joint measure. Assuming that there is a free energy potential of the form (5.9) it is the same as for the joint measure of the plus state. This is the potential constructed from (4.3) in a straightforward way. From (3.1) we see however that the conditional expectations in the infinite-volume will be different in plus-state and Dobrushin-state, because: Equality of the l.h.s. of (3.1) for different $\mu[\eta]$ implies equality of Q_x for different $\mu[\eta]$ (by varying the boundary condition $\xi_{\partial x}$). The corresponding Q_x in turn are essentially given in terms of nearest neighbor correlations and these will differ in interface states and ordered states. So, both states cannot have the same potential. This provides an example of a convergent potential constructed in a natural way that produces the wrong measure.

Finally we look at the vacuum potential with the fully occupied vacuum. We discuss again the joint measure corresponding to the ferromagnetic plus state and the Dobrushin state. If these states do exist a.s. then they also exist for the fully occupied system. So we can construct the state $\mu[\hat\eta]$, and the state $\mu[\eta]$ for typical η with the same type of boundary conditions, in both cases. Also, in both cases, we expect that $\mu[\eta_\Lambda\hat\eta^1_{\mathbb{Z}^d\setminus\Lambda}]\to\mu[\eta]$ which, in particular, implies (2.8). So the corresponding vacuum potential converges and yields the right conditional probabilities. Observe, that in a situation where a typical realization of the disorder destroys the Dobrushin state that is present for $\hat\eta^{(1)}$, a weak limit of finite volume Gibbs measure with plus/minus boundary condition $\sigma^{b.c.}$ will yield a symmetric mixture of plus and minus state. Thus, to get a correct potential, we should of course choose the corresponding $\mu[\hat\eta^{(1)}]$ to be (say) the plus state (which yields the same free energy potential as the symmetric mixture). It can be constructed e.g. with the different all-plus boundary condition $\hat\sigma^{b.c.}$. The Dobrushin state in the ordered system which will result from plus/minus boundary conditions will give a wrong potential. This explains the formulation of Theorem 2.2. where we allowed for a boundary condition $\hat\sigma^{b.c.}$ for the construction of the state $\mu[\hat\eta]$ that is different from the boundary condition $\sigma^{b.c.}$ for the construction of the state $\mu[\eta]$ for typical η.

It remains to give the

Proof of Proposition 5.1: We claim that in order that the conditional expectations be the correct ones we must have that

$$\lim_{\Delta\uparrow\mathbb{Z}^d}\sum_{A:A\subset\Delta,A\ni x}\Big(U_A^{fe}(\eta^1_x\eta_{\mathbb{Z}^d\setminus x})-U_A^{fe}(\eta^2_x\eta_{\mathbb{Z}^d\setminus x})\Big)=\log Q_x(\eta^1_x,\eta^2_x,\eta_{\mathbb{Z}^d\setminus x}) \tag{5.11}$$

for $\mathbb{P}$-a.e. η, for all η^1_x and η^2_x. This follows from the fact that the Δ-limit of (4.2) (which is assumed to exist) and (3.1) must coincide, $\mathbb{P}$-a.e., which is equivalent to

$$\int\mu_x^{\text{ann},\xi_{\partial x}}(d\tilde\eta_x)e^{-\sum_{A:A\ni x}\left(U_A^{fe}(\tilde\eta_x\eta_{Z^d\setminus x})-U_A^{fe}(\eta)\right)}=\int\mu_x^{\text{ann},\xi_{\partial x}}(d\tilde\eta_x)Q_x(\eta_x,\tilde\eta_x,\eta_{\mathbb{Z}^d\setminus x}) \tag{5.12}$$

A simple computation shows that the one-site annealed distribution is given by $\mu_x^{\text{ann},\xi_{\partial x}}(\eta_x=1)/\mu_x^{\text{ann},\xi_{\partial x}}(\eta_x=0)=\cosh(J\sum_{y\in\partial x}\eta_y\sigma_y)$. Thus, by writing (5.12) for different values of $\xi_{\partial\Lambda}$ corresponding to different values for the expression in the cosh we can conclude that (5.12) really implies (5.11). Next, we note that (5.11) implies that, for any finite $\Lambda\subset\mathbb{Z}^d$ and $x\in\Lambda$ we have that

$$\sum_{A:A\subset\Lambda,A\ni x}U_A^{fe,0}(\eta)=\log\frac{Z^0_\Lambda(\eta_x\eta_{\Lambda\setminus x})}{Z^0_\Lambda(\hat\eta_x\eta_{\Lambda\setminus x})} \tag{5.13}$$

which is seen as follows: $\mu[\eta]$ satisfies the DLR-equation for $\mathbb{P}$-a.e. η and so we have that $\mu[\eta_\Lambda\hat\eta_{\partial\Lambda}\eta_{\mathbb{Z}^d\setminus\overline\Lambda}](\sigma_\Lambda)=\mu^0_\Lambda[\eta_\Lambda](\sigma_\Lambda)$, for $\mathbb{P}$-a.e. $\eta_{\mathbb{Z}^d\setminus\overline\Lambda}$. So we have from (5.11) (putting $\eta^1_x=\eta_x$,

$\eta_x^2 = \hat{\eta}_x$)

$$\lim_{\Delta \uparrow \mathbb{Z}^d} \sum_{A: A \subset \Delta, A \ni x} U_A^{fe,0}(\eta_\Lambda \hat{\eta}_{\partial\Lambda} \eta_{\mathbb{Z}^d \setminus \overline{\Lambda}}) = \log Q_x(\eta_x, \hat{\eta}_x, \eta_{\Lambda\setminus x} \hat{\eta}_{\partial\Lambda} \eta_{\mathbb{Z}^d \setminus \overline{\Lambda}}) = \log \frac{Z_\Lambda^0(\eta_x \eta_{\Lambda\setminus x})}{Z_\Lambda^0(\hat{\eta}_x \eta_{\Lambda\setminus x})} \tag{5.14}$$

for $I\!P$-a.e. $\eta_{\mathbb{Z}^d \setminus \overline{\Lambda}}$ whenever $x \in \Lambda$. But the l.h.s. of (5.14) equals the l.h.s. of (5.13), due to the assumption on the form of the potential involving only connected A's.

Finally, from (5.13) one sees by telescoping over the sites in Λ that $\sum_{A: A \subset \Lambda} c_A^{(0)} = \sum_{A: A \subset \Lambda} U_A^{fe,0}(1_A)$ $= \log Z_\Lambda^0 / 2^{|\Lambda|}$ which, by the inclusion-exclusion formula gives the desired (5.10).$\diamondsuit$

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