# LIMITING BEHAVIOR OF RANDOM GIBBS MEASURES:

# METASTATES IN SOME DISORDERED MEAN FIELD MODELS


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**Abstract:** We present examples of random mean field spin models for which the size dependence of their Gibbs measures $\mu_{\Lambda_n}$ can be rigorously analyzed. We investigate their 'empirical metastates' $1/N\sum_{n=1}^{N}\delta_{\mu_{\Lambda_n}}$, introduced by Newman and Stein, along the sequence of finite volumes $\Lambda_n=\{1,\dots,n\}$. The empirical metastate is shown *not* to converge in our examples if the realization of the disorder is fixed. This phenomenon leads us to consider the *distributions* w.r.t disorder of the empirical metastates for which we show convergence and give explicit limiting expressions.

## I. Introduction

It is the aim of this note to discuss rigorous results on the *size dependence* of random spin systems in two examples of well known mean field systems; these will be the Curie Weiss Random Field Ising Model (CWRFIM) and the Hopfield model. We will discuss weak cluster points of Gibbs measures and 'Metastates'; for the introduction and motivation of the latter we refer the reader to the article of Newman and Stein in this volume. The motivation for studying these models is in fact to give rigorous examples of the treatment of size dependence in terms of metastates. We will only sketch the proofs of the results presented here; complete estimates can be found in [K2].

In our first example, the *Curie Weiss Random Field Ising Model* (CWRFIM), we will write

$$\mu_N(\eta)[(\sigma_i)_{i=1,\dots,N}] = \frac{1}{Norm.} \exp\left(\frac{\beta}{2N}\sum_{1\le i,j\le N}\sigma_i\sigma_j + \beta\sum_{1\le i\le N}\eta_i\sigma_i\right) \tag{1.1}$$

for the *finite volume Gibbs measures* in the volume $\{1,\dots,N\}$. $\sigma_i = \pm 1$ are Ising spins. For simplicity we take the quenched disorder variables $\eta_i$ as i.i.d. variables with $I\!P[\eta_i = \pm\epsilon] = \frac{1}{2}$. As usual the $\mu_N(\eta)$ can also be viewed as measures on the infinite spin space by tensoring with arbitrary product measures for the spins at sites $i > N$.

In our second example, the *Hopfield Model with finite number $M$ of patterns*, the finite volume Gibbs measure are denoted by

$$\mu_N(\xi)[(\sigma_i)_{i=1,\dots,N}] = \frac{1}{Norm.} \exp\left(\frac{\beta}{2N}\sum_{1\le i,j\le N}\sum_{1\le\nu\le M}\xi_i^\nu\xi_j^\nu\sigma_i\sigma_j\right) \tag{1.2}$$

The 'disorder' enters through the patterns $\xi^\mu = (\xi_i^\mu)_{i\in I\!N}$ with i.i.d. bits $I\!P[\xi_i^\mu = \pm 1] = \frac{1}{2}$.

The advantage of these mean field models is that they allow rigorously to make sense out of an approximate extreme decomposition of the form

$$\mu_N(\eta) \approx \sum_m p_N^m(\eta)\mu_\infty^m(\eta) \tag{1.3}$$

Here $\eta$ is a generic notation for the quenched disorder variable, $\mu_\infty^m(\eta)$ are the 'extremal infinite volume Gibbs measures' describing the $m$' th phase, and $p_N^m(\eta)$ are the random weights whose large $N$-behavior contains the phenomenon of size dependence. The obtained estimates for (1.3) can then be used to control the large $N$-behavior of the empirical metastate

$$\kappa_N(\eta) = \frac{1}{N}\sum_{n=1}^N \delta_{\mu_N(\eta)} \tag{1.4}$$

When dealing with convergence and approximations of the type (1.3) we have to be precise about the topology. As in [AW] and in the articles of Newman and Stein, the topologies used on the three different levels of: spins, states (probability measures on the infinite volume spin space) and metastates (probability measures on the states) are: The product topology for the spin space; the corresponding weak topology for the states; and the inherited weak topology on the metastates. The latter means that convergence is checked on functions on states $\mu$ of the form

$$F(\mu) = \tilde{F}\left(\mu(f_1), \dots, \mu(f_l)\right) \tag{1.5}$$

where $\tilde{F} : I\!R^l \to I\!R$ is a polynomial; $f_1, \dots, f_l$ are local functions on $\Omega = \{1, -1\}^{I\!N}$.

An important question that can be asked about the empirical metastate is: Does $\kappa_N(\eta)$, as defined in (1.4) with the natural sequence of volumes $\Lambda_n = \{1, \dots, n\}$, converge for fixed realization $\eta$? As we will see below, the answer is no in our examples; instead we can characterize the large $N$ behavior of $\kappa_N$ in two possible ways: by (a) fixed-$\eta$ 'pathwise' approximation; by (b) showing convergence in *distribution.*

## II. The Curie Weiss Random Field Ising Model

The phase diagram of the system is well known (see [SW],[APZ]). At low temperatures $1/\beta$ and small $\epsilon$ the model is ferromagnetic, i.e. there exist two 'pure' phases, a ferromagnetic $+$ phase $\mu^+_\infty(\eta)$ and a $-$ phase $\mu^-_\infty(\eta)$, given by the infinite product measures

$$\mu^\pm_\infty(\eta)[\sigma_\Lambda = \omega_\Lambda] = \prod_{i\in\Lambda} \frac{e^{\beta(\pm m^*(\beta,\epsilon)+\eta_i)\omega_i}}{2\cosh(\beta(\pm m^*(\beta,\epsilon)+\eta_i))} \tag{2.1}$$

where $m^*(\beta,\epsilon) \geq 0$ is the largest solution of the averaged mean field equation. We restrict our interest to the interior of this two phase region of the phase diagram. Then an approximate extreme decomposition can be written as

$$\mu_N(\eta) \approx p(W_N)\mu^+_\infty(\eta) + (1 - p(W_N))\mu^-_\infty(\eta) \tag{2.2}$$

with weights given by

$$p_N(W_N) = \frac{e^{c_2(\beta)W_N}}{e^{c_2(\beta)W_N} + e^{-c_2(\beta)W_N}} \tag{2.3}$$

Their dependence on the randomness is only through the random walk

$$W_N = \sum_{1\leq i\leq N} \frac{\eta_i}{\epsilon} \tag{2.4}$$

Assuming the validity of (2.2) for the moment we would like to point out the following observations.

(i) Given a local function $f(\sigma)$ of the spins, the random variables $p(W_N)$ and $\int \mu^{\pm}_{\infty}(\eta)(d\sigma)f(\sigma)$ become asymptotically independent for large $N$. This important phenomenon of asymptotic decoupling is also generally expected to hold in lattice systems, as long as the dependence of the pure states on the underlying field describing the quenched disorder is effectively *local.*[1]

(ii) The state $\mu_n(\eta)$ is 'pure most of the times $n$'. Since $W_n$ takes values of the order of magnitude $n^{\frac{1}{2}}$ for a large fraction of times, we can use the approximation $p(W_n) \approx 1_{W_n>0}$ for the empirical metastate to obtain

$$\frac{1}{N}\sum_{1\leq n\leq N} F\left(\mu_n(\eta)\right) \approx n_N(\eta)F(\mu^{+}_{\infty}(\eta)) + (1-n_N(\eta))F(\mu^{-}_{\infty}(\eta)) \tag{2.5}$$

with $n_N(\eta) = \frac{1}{N}\#\{1\leq n\leq N | W_n > 0\}$.

(iii) According to the classical arcsine-law for the coin-tossing random walk we have then $n_N(\eta) \to^{law} n_{\infty}$.

In fact, in [K2] the following precise results are proven. Let us write $\mathcal{CP}\{\mu_N,\, N=1,2,\dots\}$ for the set of cluster points of the sequence $\mu_N$ w.r.t the weak topology.

**THEOREM 1:** *Denote by $\mu_N(\eta)$ the finite volume Gibbs measures of a CWRFIM with $\beta,\epsilon$ lying in the interior of the two phase region. Then we have with the above notations*

*(i) Weak Cluster Points: For a.e. realization of the random fields $\eta$,*

$$\begin{aligned}&\mathcal{CP}\{\mu_N(\eta),\, N=1,2,\dots\}\\ &\quad= \left\{q\mu^{+}(\eta) + (1-q)\mu^{-}(\eta) \frac{1}{q} = 1+\exp(-2c(\beta)z),\, z\in \mathbb{Z}\cup\{+\infty\}\cup\{-\infty\}\right\}\end{aligned} \tag{2.6}$$

*(iia) Empirical metastate: For a.e. $\eta$, for all continuous $F$*

$$\lim_{N\uparrow\infty}\left(\frac{1}{N}\sum_{1\leq n\leq N} F\left(\mu_n(\eta)\right) - \left(n_N(\eta)F(\mu^{+}_{\infty}(\eta)) + (1-n_N(\eta))F(\mu^{-}_{\infty}(\eta))\right)\right) = 0 \tag{2.7}$$

*(iib) Empirical metastate: law*

$$\lim_{N\uparrow\infty}\frac{1}{N}\sum_{n=1}^{N} F(\mu_n(\eta)) =^{law} n_{\infty}F\left(\mu^{+}_{\infty}(\eta)\right) + (1-n_{\infty})F\left(\mu^{-}_{\infty}(\eta)\right) \tag{2.8}$$

[1] For an example where this local dependence of the Gibbs measure on the underlying randomness becomes apparent in a lattice system, see the construction of the groundstate in the Bricmont-Kupiainen-renormalization group treatment in [BoK], [K1].

*where $n_\infty$ is a 'fresh' random variable, independent of $\eta$ on the r.h.s. with distribution $I\!P[n_\infty < x] = \frac{2}{\pi}\arcsin\sqrt{x}$.*

*(iii) Conditioned metastate (Aizenman-Wehr metastate): For a.e. $\eta$*

$$\bar{\kappa}(\eta) = \frac{1}{2}\delta_{\mu^+(\eta)} + \frac{1}{2}\delta_{\mu^-(\eta)} \tag{2.9}$$

For (i), see also [APZ]. Note that this example shows that the set of weak cluster points can be bigger than the support of the metastates. Let us finally explain statement (iii) about the metastate obtained by conditioning the joint limiting distribution of $(\eta, \mu_N(\eta))$ (see the article of Newman and Stein in this volume). Let us recall its definition: Assume that for each jointly continuous function $G(\mu,\eta)$ (w.r.t the weak topology for $\mu$ and the product topology for $\eta$) the limit

$$\lim_{N\uparrow\infty} I\!E\left[G(\mu_N(\eta),\eta)\right] = \int K(d\mu, d\eta) G(\mu,\eta) \tag{2.10}$$

exists and defines a probability measure $K(d\mu, d\eta)$. Then we denote by $\bar{\kappa}(\eta)(d\mu)$ the regular conditional probability of $K$ given $\eta$ and call it the *conditioned metastate*. It is thus defined by the equation $\int K(d\mu, d\eta) G(\mu,\eta) = I\!E[\bar{\kappa}(\eta)(d\mu) G(\mu,\eta)]$. Now, the statement (iii) in the CWRFIM is explained by the approximations

$$\begin{aligned} I\!E\left[G(\mu_N(\eta),\eta)\right] &\approx I\!E\left[G(p(W_N)\mu^+_\infty(\eta) + (1-p(W_N))\mu^-_\infty(\eta),\eta)\right] \\ &\approx \frac{1}{2} I\!E G(\mu^+_\infty(\eta),\eta) + \frac{1}{2} I\!E G(\mu^-_\infty(\eta),\eta) \end{aligned} \tag{2.11}$$

where the last approximation uses the asymptotic decoupling of the weights and the dependence of the function $G$ other than through the weights.

Let us remark that we expect the non-convergence of the empirical metastate for fixed realization to occur also in the lattice random field Ising model if we use a sequence of nested boxes $(\Lambda_n)_{n=1,2,\ldots}$ containing $|\Lambda_n| \sim n^d$ spins. Then also an ansatz of the form (2.2) (and consequently (2.5)) is expected if we replace $W_n$ in these formulas by $\sum_{x\in\Lambda_n} \eta_x \equiv W_{|\Lambda_n|}$. Assuming this we investigate the variance

$$I\!E\left[n_N^2\right] - I\!E\left[n_N\right]^2 = \frac{1}{N^2} \sum_{1\le n,m\le N} I\!E[\operatorname{sign}(W_{|\Lambda_n|}) \operatorname{sign}(W_{|\Lambda_m|})] \tag{2.12}$$

In fact, (2.12) remains bounded below against zero when $N\uparrow\infty$, for polynomially growing sequences $|\Lambda_n|$. (This follows easily from $I\!E[\operatorname{sign}(W_N)\operatorname{sign}(W_M)] \approx$ *const* $\sqrt{\frac{N}{M-N}}$ for fixed $\frac{N}{M} << 1$ when $N, M\uparrow\infty$ .) But, if $\lim_{N\uparrow\infty} n_N$ existed for fixed realization it would have to be a.s. constant, being a tail variable. Consequently, this would imply the non-convergence of the empirical metastate for fixed realization.

## III. The Hopfield model with finite number of states

Since the Hopfield model is treated very intensively in this volume (see in particular the articles by Bovier and Gayrard, resp. Talagrand), we will only be very brief here.

The thermodynamics for finite number of patterns is very well known. The role of the infinite volume Gibbs measures is now played by the $M$ symmetric mixtures, the Mattis states

$$\mu_\infty^\nu(\xi) = \frac{1}{2}\left(\mu_\infty^{\nu+}(\xi) + \mu_\infty^{\nu-}(\xi)\right) \tag{3.1}$$

where

$$\mu_\infty^{\nu\pm}(\xi)[\sigma_\Lambda = \omega_\Lambda] = \prod_{i\in\Lambda} \frac{e^{\pm\beta m^* \xi_i^\nu \omega_i}}{2\cosh(\beta m^*)}$$

Here $m^*$ is the solution of the (ordinary) Curie Weiss equation. The asymptotic extremal decomposition becomes

$$\mu_N(\xi) \approx \sum_{1\leq\nu\leq M} p^\nu(N^{-\frac{1}{2}} b_N(\xi))\mu_\infty^\nu(\xi) \tag{3.2}$$

with the random walk

$$b_N^{\mu\nu}(\xi) = \sum_{i=1}^N (\xi_i^\mu \xi_i^\nu - \delta^{\mu\nu}) \tag{3.3}$$

It takes values $b_N(\xi) \in \mathcal{A} = \{M \times M$ symm. matrices with vanishing diagonal $\}$. The weights are then obtained through the function $p : \mathcal{A} \to \mathcal{S} = \{M -$ dim. prob. vectors$\}$ given by

$$\begin{aligned} p^\nu(V) &= \frac{\tilde{p}^\nu(V)}{\sum_\mu \tilde{p}^\mu(V)} \\ \tilde{p}^\nu(V) &= \exp\left(c(\beta)(V^2)^{\nu\nu}\right) \end{aligned} \tag{3.4}$$

with the constant $c(\beta) = \frac{\beta m^*}{2(1-\beta(1-m^*)^2)}$. We remark that, for $M \geq 3$ (which we assume to avoid trivialities), the mapping $p$ is onto (see [K2]). Let us define

$$\tilde{\kappa}_N(\xi) = \frac{1}{N}\sum_{n=1}^N \delta_{\sum_{\nu=1}^M p^\nu\left(\frac{b_n(\xi)}{\sqrt{n}}\right)\mu_\infty^\nu(\xi)} \tag{3.5}$$

To describe the limit of the law of the empirical metastate we also introduce a Brownian motion $W_t$ in $\mathcal{A}$; it is simply obtained by substituting independent standard one dimensional Brownian motions for the upper off-diagonal elements. Then the analogue of the theorem for the CWRFIM reads

**THEOREM 2:** *Denote by $\mu_N(\xi)$ the finite volume Gibbs measures of a Hopfield model with $M$ patterns at inverse temperature $\beta > 1$. Then we have with the above notations*

*(i) Weak cluster points: For a.e. $\xi$*

$$\mathcal{CP}\{\mu_N(\xi),\, N = 1,2,\ldots\} = \left\{ \sum_{1\leq\nu\leq M} q^\nu \mu_\infty^\nu(\xi),\ (q^\nu)_{\nu=1,\ldots,M} \in \mathcal{S} \right\} \tag{3.6}$$

*(iia) Empirical metastate: For a.e. $\xi$*

$$\lim_{N\uparrow\infty} \left( \int \kappa_N(\xi)(d\mu) F(\mu) - \int \tilde{\kappa}_N(\xi)(d\mu) F(\mu) \right) = 0 \tag{3.7}$$

*(iib) Empirical metastate: law*

$$\lim_{N\uparrow\infty} \frac{1}{N} \sum_{n=1}^{N} F(\mu_n(\xi)) =^{law} \int_0^1 dt F \left( \sum_{\nu=1}^{M} p^\nu \left( \frac{W_t}{\sqrt{t}} \right) \mu_\infty^\nu(\xi) \right) \tag{3.8}$$

*where $W_t$ is Brownian motion in $\mathcal{A}$, indep. of $\xi$*

*(iii) Conditioned metastate: For a.e. $\xi$*

$$\bar{\kappa}(\xi)(F) = \mathbb{E}_g F \left( \sum_{\nu=1}^{M} p^\nu (g)\, \mu_\infty^\nu(\xi) \right) \tag{3.9}$$

*where $g$ is a Normal Gaussian in $\mathcal{A}$, indep. of $\xi$*

Again, the empirical metastate does not convergence for fixed realization. In contrast to the previous example, the metastates give mass to *all mixtures* of Gibbs measures, not only to the extremal ones.

The volume dependence of the Gibbs measures when the number of patterns $N(M)$ goes to infinity will be treated in a forthcoming paper.

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## References:


[APZ] J.M.G. Amaro de Matos, A.E. Patrick, V.A. Zagrebnov, Random Infinite-Volume Gibbs States for the Curie-Weiss Random Field Ising Model., J.Stat.Phys **66**, 139-164 (1992)

[AP] J.M.G. Amaro de Matos, J.F. Perez, Fluctuations in the Curie-Weiss Version of the Random Field Ising Model., J.Stat.Phys **62**, 587-608 (1990)

[AW] M. Aizenman, J. Wehr, Rounding Effects of Quenched Randomness on First-Order Phase Transitions, Comm.Math.Phys **130**, 489-528 (1990)

[BG1] A. Bovier, V. Gayrard, The retrieval phase of the Hopfield model: A rigorous analysis of the overlap distribution, to appear in Prob.Theor.Rel.Fields (1995)

[BG2] A. Bovier, V. Gayrard, Hopfield models as a generalized random mean field model, WIAS preprint 253, Berlin (1996), to appear in 'Mathematics of spin glasses and neural networks', A. Bovier and P. Picco, eds., 'Progress in Probability', Birkhäuser (1997)

[BGP1] A. Bovier, V. Gayrard, P. Picco, Gibbs states of the Hopfield model in the regime of perfect memory, Prob.Theor.Rel.Fields **100**,329-363, (1994)

[BGP2] Gibbs states of the Hopfield model with extensively many patterns, J.Stat.Phys. **79**, 395-414 (1995)

[BoK] A. Bovier, C. Külske, A rigorous renormalization group method for interfaces in random media, Rev.Math.Phys. **6** No.3, 413-496 (1994)

[Co] F. Comets, Large Deviation Estimates for a Conditional Probability Distribution. Applications to Random Interaction Gibbs Measures, Prob.Th.Rel.Fields **80**, 407-432 (1989)

[Fe] W. Feller, An Introduction to Probability Theory and its Applications, Wiley, New York, London, Sidney, (1966)

[Gen] B. Gentz, An almost sure Central Limit Theorem for the overlap parameters in the Hopfield model, preprint Universität Zürich, to be published in Stoch.Proc.Appl. (1996)

[Geo] H.O. Georgii, Gibbs measures and phase transitions, Studies in mathematics, vol. 9 (de Gruyter, Berlin, New York, 1988)

[Ho] J.J. Hopfield, Neural networks and physical systems with emergent collective computational abilities, Proc.Natl.Acad.Sci. USA **79**, 2554-2558 (1982)

[K1] C. Külske, Ph.D. Thesis, Ruhr-Universität Bochum (1993)

[K2] C. Külske, Metastates in disordered mean field models: Random field and Hopfield models, WIAS preprint 1996, subm. to J.Stat.Phys.

[LPS] J.T. Lewis, C.E. Pfister and W.G. Sullivan, Entropy, Concentration of Probability and Conditional Limit Theorems, Markov Proc. Rel. Fields **1**, 319-386 (1995)

[N] C.M. Newman, Topics in Disordered Systems, to appear in 'ETH Lecture Notes Series', Birkhäuser (1996)

[NS1] C.M. Newman, D.L. Stein, Chaotic Size Dependence in Spin Glasses, in Cellular Automata and Cooperative Systems, Boccara, Goles, Martinez, Picco (Eds.), Nato ASI Series C Vol. 396, Kluwer, Dordrecht (1993)

[NS2] C.M. Newman, D.L. Stein, Non-Mean Field Behavior in realistic Spin glasses, Phys.Rev.Lett. **76**, No 3, 515-519 (1996)

[NS3] C.M. Newman, D.L. Stein, Spatial Inhomogeneity and thermodynamic chaos, Phys.Rev.Lett. **76**, No 25, 4821-4825 (1996)

[P] G. Parisi, Recent rigorous results support the predictions of spontaneously broken replica symmetry for realistic spin glass, preprint, March, 1996. Available as cond-mat preprint 9603101 at http://www.sissa.it

[Se] T. Seppäläinen, Entropy, limit theorems, and variational principles for disordered lattice systems, Commun.Math.Phys **171**,233-277 (1995)

[SW] S.R. Salinas, W.F. Wreszinski, On the Mean-Field Ising Model in a Random External Field, J.Stat.Phys **41**, 299-313 (1985)