ON THE GIBBSIAN NATURE OF THE RANDOM FIELD KAC MODEL UNDER BLOCK-AVERAGING *

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Abstract: We consider the Kac-Ising model in an arbitrary configuration of local magnetic fields $\eta = (\eta_i)_{i\in\mathbb{Z}^d}$, in any dimension d, at any inverse temperature. We investigate the Gibbs properties of the 'renormalized' infinite volume measures obtained by block averaging any of the Gibbs-measures corresponding to fixed η, with block-length small enough compared to the range of the Kac-interaction. We show that these measures are Gibbs measures for the same renormalized interaction potential. This potential depends locally on the field configuration η and decays exponentially, uniformly in η, for which we give explicit bounds.

Key Words: Kac-model, random field model, Gibbs-measures, renormalization group transformations

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I. Introduction

The study of models with Kac-type (= long range) potentials is a rich and fruitful subject in equilibrium statistical mechanics. Kac-models depend on a parameter γ describing the inverse range of the interaction. They were introduced by Kac [1963] to give a microscopic model in which the van der Waals theory of phase transitions could be understood. In fact, the famous Lebowitz-Penrose theorem [LP] states that, for a classical particle system with a Kac-pair-interaction, the free energy density converges, in the limit $\gamma \downarrow 0$, to the convex envelope of the mean field free energy.

In recent years there has been new interest in the study of Kac lattice-spin models (see e.g. [COP],[BBP],[CP],[BZ1],[BP]). The challenge in this direction of research is to understand these models on the level of Gibbs-measures, and not only on the level of thermodynamic potentials, for small but finite Kac-parameter γ. Even the proof of low-temperature ordering in the Kac-Ising model in more than one dimensions in zero field, at temperatures uniform as $\gamma \downarrow 0$, was only given relatively recently (independently by [CP],[BZ1]). Steps in the direction of a treatment of not necessarily symmetric long-range models are under way ([BZ3]). New behavior appears when Kac-versions of models with disorder are investigated. So far, for random models there are rigorous results about the structure of the low-temperature Gibbs measures only in one dimension. However, even here adding randomness can influence the behavior of the system in an interesting way (see [BGPi] for the Hopfield-Kac model, see [COPi] for the random field Kac-model).

It is a common step in the analysis of lattice Kac-models to try to describe the system on the level of local averages of the order parameter in blocks of a scale $l \ll \frac{1}{\gamma}$. An analogous coarse-graining from a continuous-particle system with Kac-potential to a lattice-spin system was used in the beautiful paper [LMP] to show the existence of a gas-liquid phase transition (with the distinct phases characterized by different densities).

Such a blocking transformation can be viewed as a 'renormalization group transformation' and be immediately investigated on the infinite lattice, too. Already from an abstract point of view, it is then a natural question to ask whether the resulting image measures will be Gibbsian, having in mind the numerous examples of non-Gibbsian measures emerging in seemingly innocent places in lattice spin models. In particular we remind the reader that, in short range models, one of the most prominent examples of non-Gibbsian measures is the Ising model in zero field under block averaging, at low temperature. (This was proved by [EFS] in their 'big paper', see Theorem 4.6 therein.) For a general overview on the problem of non-Gibbsian measures we also refer to the standard reference [EFS]. For more recent developments, see the review articles [E],[F] and [DS],[BKL],[MRSM]. In the case of random system, the additional question comes up to understand the interplay with the disorder variables, and see whether the resulting interactions, when they exist, are local functions of these variables, too. (For an analysis of a class of different examples of non-Gibbsian, but weakly Gibbsian measures arising from disordered systems, see [K5],[K6],[EMK].)

After the blocking is done, the situation should be easier, but it can still be highly nontrivial to control the phase structure of the blocked measure. We will not discuss this step here. It should however be clear that it can be very useful from a technical point of view to have at hand a renormalized Hamiltonian with precise estimates on the decay of the potential. So, the purpose of this note, is both (1) to present a nicely behaved coarse-graining example of a disordered system for Gibbs-theory, and also (2) provide concrete information on the given model that can be explicitly used in a later analysis. Moreover, the technicalities are relatively simple, so our treatment of the model can also serve as a pedagogical and self-contained example that shows what ingredients are needed to prove such a result.

Let us now define the model and state our results. Consider the Kac-Ising model in an arbitrary external magnetic field configuration $\eta = (\eta_i)_{i\in \mathbb{Z}^d}$. The formal Hamiltonian is

$$H[\eta](\sigma) = -\frac{\beta}{2}\sum_{i,j} J_\gamma(i-j)\sigma_i\sigma_j - \beta\sum_i \eta_i\sigma_i \tag{1.1}$$

The spin variables $\sigma = (\sigma_i)_{i\in\mathbb{Z}^d}$ take values in $\{-1,1\}^{\mathbb{Z}^d}$. We consider this formal Hamiltonian for a fixed value of the inverse temperature β and the Kac-interaction-parameter $0 < \gamma \leq 1$ describing the inverse range of the interaction. The two-spin interaction is given by $J_\gamma(i) = \gamma^d J_1(\gamma i)$ where we restrict ourselves to the simplest choice for the Kac-interaction being an indicator function $J_1(i) = c_d 1_{|i|\leq 1}$, where $|i|$ denotes the sup-norm on $\mathbb{R}^d$. $c_d = 2^{-d}$ is the normalization that is chosen such that $\int J_\gamma(x)dx = 1$ for all γ, so that the strength of the interaction of a fixed spin with the others is of the order β, independently of γ.

The aim of the paper is to study the 'renormalization group map' given by l-block-averaging which is defined as follows. Partition the lattice $\mathbb{Z}^d$ into blocks of sidelength l. Each of these blocks will be labelled by an index x, where we identify x with a coordinate vector in $\mathbb{Z}^d$. Then the block-average map is just

$$(\sigma_i)_{i\in x} \mapsto m_x\left((\sigma_i)_{i\in x}\right) = \frac{1}{l^d}\sum_{i\in x}\sigma_i \tag{1.2}$$

from $\{-1,1\}^{l^d} \to \{-1,-1+\frac{2}{l^d},\dots,1\}$. Following common notation, the last sum is over those sites i in the original lattice that lie inside a block with label x on the coarse-grained lattice. We also write $\sigma_x = (\sigma_i)_{i\in x}$ (and $\eta_x = (\eta_i)_{i\in x}$) to denote the collection of Ising spins (resp. external fields) in the block x. Let us denote by the symbol T_l the corresponding map on the infinite volume configuration spaces, obtained by application of (1.2) independently over the blocks.

As usual in Kac-models, it is then straightforward to extract a main part for the corresponding hypothetical coarse-grained energy function (say, in finite volume). What is less clear is the behavior of the error terms (the 'blocking error') and whether they give rise to a nicely absolutely convergent potential. In this context we have the following explicit result.

Theorem 1: *Assume that $\eta \in I\!R^{\mathbf{Z}^d}$ is an arbitrary external field configuration and $\mu[\eta]$ is any of the infinite volume Gibbs-measure for the corresponding d-dimensional Kac-Hamiltonian (1.1). Suppose that the block length $l \in \{2,3,4,\dots\}$ is less or equal than the range of the interaction $\frac{1}{\gamma}$ and, moreover, that the parameters l, β, γ are such that the 'expansion parameter'*

$$\lambda(\beta,\gamma,l) := \sum_{x\in\mathbf{Z}^d}\left(\exp\Bigl(\beta\sum_{\substack{i,j:\\ i\in x,\, j\in 0}}|J_\gamma(i-j)-J_\gamma(lx)|\Bigr)-1\right) \tag{1.3}$$

is less or equal than $\lambda^* \approx 0.110909...$.

Then, the l-coarse-grained measure $T_l\mu[\eta]$ is a Gibbs-measure for an Hamiltonian with exponentially decaying interactions.

This Hamiltonian has the form

$$\begin{aligned} &H^{\mathrm{ren}}[\eta]\left((m_x)_{x\in\mathbf{Z}^d}\right)\\ &=\beta'\left(\frac{1}{4}\sum_{x,y\in\mathbf{Z}^d}J_{\gamma l}(x-y)(m_x-m_y)^2+\sum_{x\in\mathbf{Z}^d}f_{\beta,l}[\eta_x](m_x)\right)-\sum_{A:A\subset Z^d}U_A(\eta_A,m_A)\end{aligned} \tag{1.4}$$

Here $\beta' \equiv \beta l^d$ is the renormalized inverse temperature. The single site potentials are given by the 'finite block free-energies'

$$f_{\beta,l}[\eta_x](m_x) = -\frac{1}{\beta l^d}\log\mu^0[\eta_x]\left(m_x(\sigma_x)=m_x\right)-\frac{m_x^2}{2}(1+\epsilon_{\gamma l}) \tag{1.5}$$

where $\mu^0[\eta_x](\sigma_x=\omega_x)=\prod_{i\in x}\frac{\exp(\beta\eta_i\omega_i)}{2\cosh(\beta\eta_i)}$ is the product measure obtained by putting the Kac-coupling J equal to zero, and $1+\epsilon_{\gamma l}=\sum_{z\in\mathbf{Z}^d}J_{\gamma l}(z)$ is close to one for γl small.

The multi-body potentials U_A stem from the expansion of the 'blocking error'. They are non-zero only for $|A|\geq 2$ and satisfy the bound

$$\sum_{A:A\ni x}|U_A(\eta_A,m_A)|\left(\frac{\lambda^*}{\lambda(\beta,\gamma,l)}\right)^{|A|-1}\leq a^*\approx 0.633 \tag{1.6}$$

independently of x and of the external fields η.

Remark: Note that we did not make any assumption on the random field configuration. Of course, in the 'true' random field model, one is interested in the behavior of the system for *typical* configurations η that are drawn from an i.i.d. distribution on the infinite lattice. (See [BK],[AW] for mathematical results on the random field Ising model, see [K3] for the continuous spin version.) The Gibbs measures of untypical configurations can of course have very different properties. Even though, our theorem states that the map from Hamiltonian to renormalized Hamiltonian stays simple. As long as there is smallness of the parameter $\lambda(l,\beta,\gamma)$ it is irrelevant whether the original system undergoes a phase transition or not.

Remark: The condition on the parameters essentially means that $\beta'\gamma l$ has to be small enough, see (2.28).

Remark: The first two terms in the formula are what one expects to describe the leading order behavior of the Kac-model. The first term favors configurations of constant block-magnetisations m_x, with the scaled range of interaction γl. The single site potentials given by $f_{\beta,l}[\eta_x](m_x)$ favor configurations close to its minima, which are determined by η_x, the value of the external fields on the block. For vanishing external fields, the potential converges with $l\uparrow\infty$ to the free-energy function of the Curie-Weiss model whose minima are the (one or two) possible values of the magnetization. More generally, for an i.i.d. random η the functions $f_{\beta,l}[\eta_x](m_x)$ converge a.s. to the non-random free-energy function of the Curie Weiss random field model. About this simple model very explicit information is known, see e.g. [AP],[APZ],[K1],[K2].

So we see that we are here in a particularly nice situation where the renormalized Hamiltonian is given by a main part obtained by a straightforward computation and corrections that are quickly decaying and explicitly controlled. Let us just mention some results of an analogous character in different lattice models. [BCO] were able to treat the entire high-temperature phase of the l-blocked Ising model with Gaussian scaling by elaborate expansions and provided explicit control on the non-Gaussian terms of the resulting potential when $l\uparrow\infty$. In [K3], [K4] single-site coarse-grainings from random continuous spin-systems to discrete ones (that turned out to be Gibbsian) were used to analyse the phase-structure. It might seem somewhat surprising that the construction of the full renormalized potential for a lattice Kac system was not formally investigated before; but say in [BZ1] the problem was bypassed by different methods and controlling the 'blocking-error' by uniform bounds.

The proof of the theorem, given in the next chapter, relies on a suitable polymer expansion of the 'blocking error'. We do the coarse graining of the original model in finite volume for any given fixed Ising-boundary condition. We show convergence of the expanded blocking terms, uniformly in the volume, the boundary condition, and the configuration of the external fields (See Proposition 2.1). For this we employ a general explicit convergence criterion for long-range graphs on the lattice, given in the appendix, which is responsible for the numerical constants occuring in the Theorem. Uniformity in the volume, for all boundary conditions, then implies the infinite volume result for all Gibbs measures, with the same bounds, by the general Proposition 2.2.

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II. Proof of the theorem: Expansion of the blocking-error

The proof of the theorem relies on the following finite volume result.

Denote the finite volume Gibbs measures of the Kac-model (= original system) with boundary condition $\bar\sigma$ and field configuration η in the volume $\Lambda\subset\mathbb{Z}^d$ by

$$\mu_\Lambda^{\bar\sigma_{\mathbb{Z}^d\setminus\Lambda}}[\eta_\Lambda](f)=\frac{\sum_{\sigma_\Lambda} f(\sigma_\Lambda\bar\sigma_{\mathbb{Z}^d\setminus\Lambda})e^{-H_\Lambda[\eta_\Lambda](\sigma_\Lambda|\bar\sigma_{\mathbb{Z}^d\setminus\Lambda})}}{\sum_{\sigma_\Lambda} e^{-H_\Lambda[\eta_\Lambda](\sigma_\Lambda|\bar\sigma_{\mathbb{Z}^d\setminus\Lambda})}} \tag{2.1}$$

where f is any spin observable and $H_\Lambda[\eta_\Lambda](\sigma_\Lambda|\bar\sigma_{\mathbb{Z}^d\setminus\Lambda})$ is the restriction of the infinite volume Hamiltonian (1.1) to Λ. As usual, it is obtained by keeping only pairs $\{i,j\}$ in the first sum and i in the second sum that are not contained in the complement of Λ, and substituting $\bar\sigma_i$ for sites i outside Λ.

Proposition 2.1: *Assume that l,β,γ are as in the hypothesis of Theorem 1, that is $l\leq\frac{1}{\gamma}$, $l\in\{2,3,4,\dots\}$ and $\lambda(\beta,\gamma,l)=\sum_{x\in\mathbb{Z}^d}\left(e^{\beta\sum_{i,j:i\in x,j\in 0}|J_\gamma(i-j)-J_\gamma(lx)|}-1\right)\leq\lambda^*\approx 0.110909\dots$*

Let $V\subset\mathbb{Z}^d$ denote a finite volume in the coarse-grained lattice and $\Lambda=\{i\in\mathbb{Z}^d|x(i)\in V\}$ be the corresponding set of sites in the original lattice.

Then, the corresponding finite volume coarse-grained measure with boundary condition $\bar\sigma_{\mathbb{Z}^d\setminus\Lambda}$ has the representation

$$\mu_\Lambda^{\bar\sigma_{Z^d\setminus\Lambda}}[\eta_\Lambda](m_V(\sigma_\Lambda)=m_V)=\frac{\exp\left(-H_V^{\bar\sigma_{Z^d\setminus\Lambda},\mathrm{ren}}[\eta_\Lambda](m_V)\right)}{\sum_{\tilde m_V}\exp\left(-H_V^{\bar\sigma_{Z^d\setminus\Lambda},\mathrm{ren}}[\eta_\Lambda](\tilde m_V)\right)} \tag{2.2}$$

Here the 'finite-volume renormalized Hamiltonian' is given by

$$\begin{aligned}&H_V^{\bar\sigma_{Z^d\setminus\Lambda},\mathrm{ren}}[\eta_\Lambda](m_V)\\&=\beta'\left(\frac{1}{4}\sum_{x,y\in V}J_{\gamma l}(x-y)(m_x-m_y)^2+\sum_{x\in\mathbb{Z}^d}f_{\beta,l}^{\bar\sigma_{Z^d\setminus\Lambda}}[\eta_x](m_x)\right)-\sum_{A:A\subset V}U_A^{\bar\sigma_{Z^d\setminus\Lambda}}(\eta_A,m_A)\end{aligned} \tag{2.3}$$

with renormalized inverse temperature $\beta' = \beta l^d$ and 'finite block free-energies' incorporating the finite volume corrections given by

$$f_{\beta,l}^{\bar\sigma_{Z^d\setminus\Lambda}}[\eta_x](m_x) = -\frac{1}{\beta l^d}\log \mu_x^{0,\bar\sigma_{Z^d\setminus\Lambda}}[\eta_x]\Bigl(m_x(\sigma_x)=m_x\Bigr) - \frac{m_x^2}{2}\sum_{y\in V} J_{\gamma l}(x-y) \tag{2.4}$$

where

$$\mu_x^{0,\bar\sigma_{Z^d\setminus\Lambda}}[\eta_x]\bigl(\sigma_x=\omega_x\bigr) := \prod_{i\in x}\frac{\exp\Bigl(\beta\bigl(\eta_i+\sum_{j\in\mathbf{Z}^d\setminus\Lambda}J_\gamma(i-j)\bar\sigma_j\bigr)\ \omega_i\Bigr)}{2\cosh\Bigl(\beta\bigl(\eta_i+\sum_{j\in\mathbf{Z}^d\setminus\Lambda}J_\gamma(i-j)\bar\sigma_j\bigr)\Bigr)} \tag{2.5}$$

is the product measure obtained by putting the Kac-coupling J equal to zero inside Λ, but keeping the couplings to the boundary.

The multi-body potentials $U_A^{\bar\sigma_{Z^d\setminus\Lambda}}$ are non-zero only for $|A|\geq 2$. They are independent on the boundary condition $\bar\sigma_{\mathbf{Z}^d\setminus\Lambda}$ for $d(A,V^c)>\frac{1}{\gamma l}$ and we have the bound

$$\sum_{A:A\ni x}\Bigl|U_A^{\bar\sigma_{Z^d\setminus\Lambda}}(\eta_A,m_A)\Bigr|\left(\frac{\lambda^*}{\lambda(\beta,\gamma,l)}\right)^{|A|-1}\leq a^*\approx 0.633 \tag{2.6}$$

uniformly in x, the boundary condition $\bar\sigma_{\mathbf{Z}^d\setminus\Lambda}$ and in the external fields η.

Remark: Apart from boundary-corrections the finite-volume coarse-grained Hamiltonian is of the desired form given in Theorem 1. Note that the interaction term is only between m_x's for sites that lie in the volume V. The main influence of the Ising-boundary condition is in the f-terms acting as local potentials on the coarse-grained variables. E.g., for mainly plus boundary Ising spins this potential will favor positive values of m_x, for x close to the boundary.

Proof: It is convenient to collect the linear parts of the RF-Kac-Hamiltonian including the boundary terms and define measures that just contain these parts. This is the reason for defining the measures (2.5). We denote more generally by $\mu_\Lambda^{0,\bar\sigma_{Z^d\setminus\Lambda}}[\eta_\Lambda](\sigma_\Lambda=\omega_\Lambda)$ the corresponding product measure on the Ising configurations in the whole of Λ. Then we can rewrite the expectation of any observable f w.r.t. the finite volume Gibbs-measures in the volume Λ with boundary condition $\bar\sigma_{\mathbf{Z}^d\setminus\Lambda}$ in the form

$$\mu_\Lambda^{\bar\sigma_{Z^d\setminus\Lambda}}[\eta_\Lambda](f) = \frac{\int\mu_\Lambda^{0,\bar\sigma_{Z^d\setminus\Lambda}}[\eta_\Lambda](d\sigma_\Lambda)f(\sigma_\Lambda\bar\sigma_{\mathbf{Z}^d\setminus\Lambda})e^{\frac{\beta}{2}\sum_{i,j\in\Lambda}J_\gamma(i-j)\sigma_i\sigma_j}}{\int\mu_\Lambda^{0,\bar\sigma_{Z^d\setminus\Lambda}}[\eta_\Lambda](d\sigma_\Lambda)e^{\frac{\beta}{2}\sum_{i,j\in\Lambda}J_\gamma(i-j)\sigma_i\sigma_j}} \tag{2.7}$$

Here we have achieved that external fields and boundary conditions are absorbed in our new a-priori measures. We introduce non-normalized m_V-weights by the constrained expectations

$$Z_\Lambda^{\bar\sigma_{Z^d\setminus\Lambda}}[\eta_\Lambda](m_V) := \int\mu^{0,\bar\sigma_{Z^d\setminus\Lambda}}[\eta_\Lambda](d\sigma_\Lambda)1_{m_x(\sigma_x)=m_x\forall x\in V}\,e^{\frac{\beta}{2}\sum_{i,j\in\Lambda}J_\gamma(i-j)\sigma_i\sigma_j} \tag{2.8}$$

so that the desired image measure we would like to control becomes

$$\mu_\Lambda^{\bar\sigma_{Z^d\setminus\Lambda}}[\eta_\Lambda](m_V(\sigma_\Lambda)=m_V)=\frac{Z_\Lambda^{\bar\sigma_{Z^d\setminus\Lambda}}[\eta_\Lambda](m_V)}{\sum_{\tilde m_V}Z_\Lambda^{\bar\sigma_{Z^d\setminus\Lambda}}[\eta_\Lambda](\tilde m_V)} \tag{2.9}$$

Now comes the blocking. To rewrite the non-normalized weights (2.8) use the constraint to get

$$\begin{aligned}&Z_\Lambda^{\bar\sigma_{Z^d\setminus\Lambda}}[\eta_\Lambda](m_V)=e^{\frac{\beta l^{2d}}{2}\sum_{x,y\in V}J_\gamma(l(x-y))m_xm_y}\\&\times\int\mu^{0,\bar\sigma_{Z^d\setminus\Lambda}}[\eta_\Lambda](d\sigma_\Lambda)\prod_{x\in\Lambda}1_{m_x(\sigma_x)=m_x}e^{\frac{\beta}{2}\sum_{x,y\in V}\sum_{\substack{i,j:\\ i\in x,j\in y}}(J_\gamma(i-j)-J_\gamma(l(x-y)))\sigma_i\sigma_j}\end{aligned} \tag{2.10}$$

The trick is to make the last line into an expectation w.r.t. a probability measure. Write $\mu_\Lambda^{0,\bar\sigma_{Z^d\setminus\Lambda}}[\eta_\Lambda](d\sigma_\Lambda|m_V):=\prod_{x\in V}\mu_x^{0,\bar\sigma_{Z^d\setminus\Lambda}}[\eta_x](d\sigma_x|m_x)$ where the last terms denote blockwise independent probability measures on the original spins conditioned on their magnetization, i.e.

$$\int\mu_\Lambda^{0,\bar\sigma_{Z^d\setminus\Lambda}}[\eta_x](d\sigma_x|m_x)f(\sigma_x)=\frac{\int\mu^{0,\bar\sigma_{Z^d\setminus\Lambda}}[\eta_x](d\sigma_x)1_{m_x(\sigma_x)=m_x}f(\sigma_x)}{\int\mu^{0,\bar\sigma_{Z^d\setminus\Lambda}}[\eta_x](d\sigma_x)1_{m_x(\sigma_x)=m_x}} \tag{2.11}$$

We put

$$I_l^{\bar\sigma_{Z^d\setminus\Lambda}}[\eta_x](m_x):=-\frac{1}{l^d}\log\mu_x^{0,\bar\sigma_{Z^d\setminus\Lambda}}[\eta_x]\left(m_x(\sigma_x)=m_x\right)\geq 0 \tag{2.12}$$

By dropping the superscript we denote the quantity obtained by putting the boundary condition $\bar\sigma_{\mathbb{Z}^d\setminus\Lambda}$ equal to zero. Of course, for sites x sufficiently far away from the boundary of V, the boundary condition is not felt anymore, and the two quantities coincide.

This function is the first part of the free-energy-like function (2.4) occuring as single site-potential. For large l (and vanishing or random η) it becomes close to a rate function. In this way we can write the constrained weight (2.8) in the form

$$\begin{aligned}&Z_\Lambda^{\bar\sigma_{Z^d\setminus\Lambda}}[\eta_\Lambda](m_V)=e^{\frac{\beta'}{2}\sum_{x,y\in V}J_{\gamma l}(x-y)m_xm_y-l^d\sum_{x\in V}I_l^{\bar\sigma_{Z^d\setminus\Lambda}}[\eta_x](m_x)}\\&\times\int\mu_\Lambda^{0,\bar\sigma_{Z^d\setminus\Lambda}}[\eta_\Lambda](d\sigma_\Lambda|m_V)e^{\frac{\beta}{2}\sum_{x,y\in V}\sum_{\substack{i,j:\\ i\in x,j\in y}}(J_\gamma(i-j)-J_\gamma(l(x-y)))\sigma_i\sigma_j}\end{aligned} \tag{2.13}$$

The negative exponent of the exponential in the first line equals the renormalized Hamiltonian (2.3) up to the U-terms. To see this, use the equation $2m_xm_y=-(m_x-m_y)^2+m_x^2+m_y^2$ and definition (2.4). The next line of (2.13) gives corrections. Now, the whole story is that these corrections can be expressed as a convergent series of interaction potentials for the block variables. In order to do this we perform a high-temperature-type expansion and produce a polymer-partition function, with weights depending locally on the m_x's and η_x's. This is seen as follows:

We define the set of pairs on the coarse-grained lattice between which an interaction can take place, i.e.

$$\mathcal{B}_{\gamma,l}:=\{\{x,y\},x,y\in V:\exists i\in x,\exists j\in y:J_\gamma(i-j)\neq J_\gamma(l(x-y))\} \tag{2.14}$$

Note that, for $l \leq \frac{1}{\gamma}$, only interactions between different sites x, y occur. With this definition we can rewrite the blocking corrections given by the second line in (2.13) as

$$\begin{aligned} &\int \mu_{\Lambda}^{0,\bar{\sigma}_{Z^d\setminus\Lambda}}[\eta_\Lambda](d\sigma_\Lambda \big| m_\Lambda) \prod_{\{x,y\}:\{x,y\}\in\mathcal{B}_{\gamma,l}} \left(e^{\beta \sum_{i,j:i\in x, j\in y}(J_\gamma(i-j)-J_\gamma(l(x-y)))\sigma_i\sigma_j} - 1 + 1 \right) \\ &= \sum_{B:B\subset\mathcal{B}_{\gamma,l}} \int \mu_{\Lambda}^{0,\bar{\sigma}_{Z^d\setminus\Lambda}}[\eta_\Lambda](d\sigma_\Lambda \big| m_\Lambda) \prod_{\{x,y\}:\{x,y\}\in B} \left(e^{\beta \sum_{i,j:i\in x, j\in y}(J_\gamma(i-j)-J_\gamma(l(x-y)))\sigma_i\sigma_j} - 1 \right) \end{aligned} \tag{2.15}$$

For a set of bonds B we denote the corresponding vertex set by $X(B) := \{x \in V : \exists y \in V, \{x,y\} \in B\}$. The simple but crucial point is that the μ^0-integration factorizes over connected components of the graph $(X(B), B)$. This allows to do an expansion in the usual way. The interesting points being left are to get reasonable bounds to prove convergence and to keep track of the dependence on external fields and boundary condition.

More precisely, we write $B = P_1 \cup \ldots \cup P_n$ for the unique decomposition into connected components and call the P_i's *polymers*. So, a polymer is a connected subgraph of $(X(\mathcal{B}_{\gamma,l}), \mathcal{B}_{\gamma,l})$. We write $\mathcal{P} \equiv \mathcal{P}_{\gamma,l} \equiv \mathcal{P}_{\gamma,l}(V)$ for the set of all such polymers in V. There is the obvious notion of pairwise compatibility: P_1, P_2 are compatible iff $X(P_1) \cap X(P_2) = \emptyset$.

So we can continue to write the last expression as a sum over pairwise compatible families of polymers with m- dependent activities of the form

$$\begin{aligned} &\int \mu_{\Lambda}^{0,\bar{\sigma}_{Z^d\setminus\Lambda}}[\eta_\Lambda](d\sigma_\Lambda \big| m_\Lambda) e^{\frac{\beta}{2}\sum_{x,y\in V}\sum_{i,j:i\in x,j\in y}(J_\gamma(i-j)-J_\gamma(l(x-y)))\sigma_i\sigma_j} \\ &= \sum_{(P_1,\ldots,P_n)_c} \prod_{i=1}^{n} \rho_{P_i}^{\bar{\sigma}_{Z^d\setminus\Lambda}}[\eta_{X(P_i)}, m_{X(P_i)}] \end{aligned} \tag{2.16}$$

This is the formulation of a polymer partition function, of the form given in appendix (A.1). Here the polymer activity of a polymer P is given by

$$\begin{aligned} &\rho_P^{\bar{\sigma}_{Z^d\setminus\Lambda}}[\eta_{X(P)}, m_{X(P)}] \\ &= \int \mu_{\Lambda}^{0,\bar{\sigma}_{Z^d\setminus\Lambda}}[\eta_{X(P)}](d\sigma_{X(P)} \big| m_{X(P)}) \prod_{\{x,y\}:\{x,y\}\in P} \left(e^{\beta \sum_{i,j:i\in x,j\in y}(J_\gamma(i-j)-J_\gamma(l(x-y)))\sigma_i\sigma_j} - 1 \right) \end{aligned} \tag{2.17}$$

The activity depends on the coarse-grained field m and external field η only on the values for x's in the vertex-set $X(P)$. The dependence on the boundary condition $\bar{\sigma}$ is only for $X(P)$ near to the boundary.

We want to perform the corresponding cluster-expansion for the logarithm of this polymer partition function. This is nothing but the Taylor-expansion when the polymer-activities are treated as independent (complex) variables ρ_P. General information on its structure and an explicit convergence criterion is given in Proposition A.1 in the appendix. To control the expansion we need estimates on the magnitude of the polymer weights. We employ a uniform bound for the terms under the integral that is valid on each pair of cubes x, y, uniformly in the spin-configurations. Using $|e^x - 1| \leq e^{|x|} - 1$ we have

$$\begin{aligned} &\left| e^{\beta \sum_{i,j: i \in x, j \in y} (J_\gamma(i-j) - J_\gamma(l(x-y)))\sigma_i \sigma_j} - 1 \right| \\ &\leq e^{\beta \sum_{i,j: i \in x, j \in y} |J_\gamma(i-j) - J_\gamma(l(x-y))|} - 1 =: e^{-\tau_{x,y}} \end{aligned} \tag{2.18}$$

This immediately gives an estimate that doesn't depend on the integrals any more, and hence

$$\left| \rho_P^{\bar{\sigma}_{Z^d \setminus \Lambda}} [\eta_{X(P)}, m_{X(P)}] \right| \leq e^{-\sum_{b: b \in P} \tau_b}, \tag{2.19}$$

independently of the values of m, η, and $\bar{\sigma}$. In our case where the Kac-interaction is given in terms of the characteristic function $J_\gamma(|i-j|) = c_d \gamma^d 1_{|i-j| \leq \frac{1}{\gamma}}$ we have that $e^{-\tau_b} = e^{\beta l^{2d} c_d \gamma^d} - 1$, independently of b unless it is zero. Looking at the definition of the 'expansion parameter' (1.3) the Proposition A.1 now ensures convergence of the expansion under the assumption $\lambda(\beta, \gamma, l) \leq \lambda^*$.

In our case, the activities are functions of m, η, $\bar{\sigma}$, and consequently the cluster-weights are functions of them, too. Indeed, we can write the logarithm of (2.16) as a cluster-sum

$$\sum_{\mathcal{C}} \Phi_{\mathcal{C}}^{\bar{\sigma}_{Z^d \setminus \Lambda}} (\eta_{X(\mathcal{C})}, m_{X(\mathcal{C})}) \tag{2.20}$$

where the sum is over all indecomposable sets $\mathcal{C}$ of polymers. We have written $X(\mathcal{C}) = \cup_{P: P \in \mathcal{C}} X(P)$. Since the cluster-weights are just sums over terms in the Taylor-expansion, the local dependence on external field and boundary condition of the polymer-weights immediately carries over to the cluster-weights $\Phi_{\mathcal{C}}^{\bar{\sigma}_{Z^d \setminus \Lambda}} (\eta_{X(\mathcal{C})}, m_{X(\mathcal{C})})$, as indicated. These facts, are collected in Proposition A.1 in the appendix.

Finally we resum over the clusters with fixed vertex sets $X(\mathcal{C})$ to obtain the representation for the logarithm of the blocking error of the desired form

$$\begin{aligned} &\log \int \mu_\Lambda^{0, \bar{\sigma}_{Z^d \setminus \Lambda}} [\eta_\Lambda] (d\sigma_\Lambda | m_V) e^{\frac{\beta}{2} \sum_{x,y \in V} \sum_{\substack{i,j: \\ i \in x, j \in y}} (J_\gamma(i-j) - J_\gamma(l(x-y)))\sigma_i \sigma_j} \\ &= \sum_{A: A \subset V} U_A^{\bar{\sigma}_{Z^d \setminus \Lambda}} (\eta_A, m_A) \end{aligned} \tag{2.21}$$

where

$$U_A^{\bar{\sigma}_{Z^d \setminus \Lambda}} (\eta_A, m_A) = \sum_{\mathcal{C}: X(\mathcal{C}) = A} \Phi_{\mathcal{C}}^{\bar{\sigma}_{Z^d \setminus \Lambda}} (\eta_{X(\mathcal{C})}, m_{X(\mathcal{C})}) \tag{2.22}$$

From the general decay estimate on cluster-sums given in appendix (A.4) we immediately get the decay-estimate on the potentials of the form

$$\sum_{A:A\cap\{x,y\}\neq\emptyset}\left|U_A^{\bar\sigma_{Z^d\setminus\Lambda}}(\eta_A,m_A)\right|\left(\frac{\lambda^*}{\lambda}\right)^{|A|-1}\leq\sum_{\mathcal{C}:\mathcal{C}\,\mathrm{icp}\,\{\mathrm{x,y}\}}|\Phi_{\mathcal{C}}^{\bar\sigma_{Z^d\setminus\Lambda}}(\eta_{X(\mathcal{C})},m_{X(\mathcal{C})})|\left(\frac{\lambda^*}{\lambda}\right)^{|\mathcal{C}|}\leq a^* \tag{2.23}$$

for any polymer $\{x,y\}$. This, in particular, implies the desired estimate (2.6) and thus finishes the proof of proposition 2.1. $\diamondsuit$

Remark: The reader might find it instructive to write down the exact expression of the pair-interactions in the potential U. It is easy to see by summing the terms in the Taylor-expansion containing just a given polymer-weight $\rho_{\{x,y\}}$ that

$$\begin{aligned}&U_{\{x,y\}}^{\bar\sigma_{Z^d\setminus\Lambda}}(\eta_{\{x,y\}},m_{\{x,y\}})\\&=\log\int\mu_x^{0,\bar\sigma_{Z^d\setminus\Lambda}}[\eta_x](d\sigma_x\big|m_x)\int\mu_y^{0,\bar\sigma_{Z^d\setminus\Lambda}}[\eta_y](d\sigma_y\big|m_y)e^{\beta\sum_{i,j:i\in x,j\in y}(J_\gamma(i-j)-J_\gamma(l(x-y)))\sigma_i\sigma_j}\end{aligned} \tag{2.24}$$

(Here we have used the form $\rho_{\{x,y\}}=\int e^{\cdots}-1$ and that the sum of terms corresponding only to the single bond polymer-weight in the Taylor expansion of the logarithm of the partition function is $\log(1+\rho_{\{x,y\}})=\log\int e^{\cdots}$.)

We can get a uniform bound on this pair potential which is better than what would follow from (2.6) by using the uniform bound $\beta l^{2d}c_d\gamma^d$ on the modulus of the argument of the exponential under the integral in (2.24). So we have

$$\left|U_{\{x,y\}}^{\bar\sigma_{Z^d\setminus\Lambda}}(\eta_{\{x,y\}},m_{\{x,y\}})\right|\leq c_d\beta'(\gamma l)^d \tag{2.25}$$

In particular we get from this and (2.6) for the higher terms that

$$\sum_{A:A\ni x}\left|U_A^{\bar\sigma_{Z^d\setminus\Lambda}}(\eta_A,m_A)\right|\leq c_d\beta'(\gamma l)^d v_{\gamma,l}+a^*\left(\frac{\lambda(\beta,\gamma,l)}{\lambda^*}\right)^2 \tag{2.26}$$

where

$$v_{\gamma,l}:=\#\left\{x\in\mathbb{Z}^d:\exists i\in x,\exists j\in 0:J_\gamma(i-j)\neq J_\gamma(lx)\right\} \tag{2.27}$$

is the number of sites that can interact with a given site via pair-interactions at all. To check the quality of our estimates note that $\sum_{x\in V}\sum_{A\ni x}|U_A|/|A|$ is an upper bound on the modulus of the logarithm of (2.16), the 'total blocking error in Λ'. Thus, $|V|/2$ times the r.h.s. of (2.26) is an upper bound on the total blocking error. But, $c_d\beta'(\gamma l)^d v_{\gamma,l}|V|/2$ is precisely the upper bound on would get on the total blocking error by doing a uniform estimate in the argument of the exponential in the last line of (2.13) in the whole of Λ, without expansion. So, the only difference is in the higher order terms and we have lost very little by summing back the expansion.

It is a simple geometric fact that there is a dimensional dependent constant c'_d s.t. $v_{\gamma,l} \leq c'_d(\gamma l)^{-(d-1)}$ (see e.g. Lemma 2.1 [BZ1]). So we have that

$$\lambda(\beta,\gamma,l) \leq v_{\gamma,l}\left(e^{\beta' c_d (l\gamma)^d} - 1\right) \leq c'_d c_d \beta' \gamma l \times e^{\beta' c_d (l\gamma)^d} \tag{2.28}$$

using that $e^{|x|} - 1 \leq |x| e^{|x|}$. This shows that $\beta'\gamma\lambda$ has to be small enough for the expansion to work.

Remark: One may ask what happens in the case $l > \frac{1}{\gamma}$ where blocks are larger than the range of the interaction, forgetting about the smallness of γ and the motivation of taking l-averages to analyse the Kac-limit. This is of a different nature altogether. We remind the reader that, by a result of van Enter, Fernandez, Sokal, there is provably non-Gibbsianness in the usual nearest-neighbor Ising model in zero field, for all even l, at sufficiently low temperature (see Theorem 4.6 in the big paper [EFS]). Of course, at sufficiently high-temperature there will be again Gibbsianness in the Kac-model: An expansion of the couplings between neighboring blocks as indicated by the formal equation

$$\begin{aligned} e^{\frac{\beta}{2}\sum_{i,j} J_\gamma(i-j)\sigma_i\sigma_j + \beta\sum_i \eta_i\sigma_i} &= \prod_x e^{\frac{\beta}{2}\sum_{i,j\in x} J_\gamma(i-j)\sigma_i\sigma_j + \beta\sum_{i\in x}\eta_i\sigma_i} \\ &\times \prod_{\{x,y\}, x\neq y}\left(e^{\beta\sum_{i\in x, j\in y} J_\gamma(i-j)\sigma_i\sigma_j} - 1 + 1\right) \end{aligned} \tag{2.29}$$

where only neighboring x, y occur, would provide us with an exponentially decaying η-dependent potential if the 'expansion parameter' $\sum_{x\in\mathbb{Z}^d, x\neq 0}\left(e^{\beta\sum_{i,j: i\in x, j\in 0} J_\gamma(i-j)} - 1\right)$ is smaller than a suitable constant. This is seen as in the proof of our Theorem. Noting that the number of pairs of spins at sites in neighboring blocks having non-zero interaction with range $1/\gamma$ is of the order $(l^{d-1}/\gamma)^2$ this immediately implies existence of a convergent interaction potential for $\beta l^d (\gamma l)^{d-2}$ sufficiently small. For better results, more elaborate expansions would have to be done.

Now, to carry over the results of Proposition 2.1 to the infinite volume and prove the theorem, we use the following general fact about Gibbs-measures under block transformations. It says that control of the coarse-grained measure uniform in the finite volume, gives Gibbsianness with the same estimates in infinite volume.

Proposition 2.2: *Suppose that $\mu_\Lambda^{\bar\sigma_{Z^d\setminus\Lambda}}(\cdot)$ are local specifications, not necessarily translation invariant, for a lattice spin system with finite local spin-space. Fix l and suppose that we are given arbitrary local maps $\sigma_x \mapsto m_x(\sigma_x)$, for all l-blocks x. Assume that we have the finite volume Gibbs-type representation*

$$\mu_\Lambda^{\bar\sigma_{Z^d\setminus\Lambda}}(m_V(\sigma_\Lambda)=m_V) = \frac{e^{-\sum_{A\subset V}\Phi_A^{\bar\sigma_{Z^d\setminus\Lambda}}(m_V)}}{\sum_{\tilde m_V} e^{-\sum_{A\subset V}\Phi_A^{\bar\sigma_{Z^d\setminus\Lambda}}(\tilde m_V)}} \tag{2.30}$$

where $V\subset\mathbb{Z}^d$ denotes a finite volume in the coarse-grained lattice, and $\Lambda=\{i\in\mathbb{Z}^d | x(i)\in V\}$ is the corresponding set of sites in the original lattice.

Assume that the above 'finite-volume potential' $\Phi^{\bar\sigma_{Z^d\setminus\Lambda}}$ has the following properties.

(i) It is absolutely uniformly summable, for all fixed boundary conditions $\bar\sigma$, uniformly in the volume Λ, i.e. that we have for all $x\in\mathbb{Z}^d$

$$\sup_\Lambda \sum_{A:A\ni x} \|\Phi_A^{\bar\sigma_{Z^d\setminus\Lambda}}\|_\infty < \infty \tag{2.31}$$

(ii) It converges to an infinite volume potential

$$\lim_{\Lambda\uparrow\mathbb{Z}^d} \Phi_A^{\bar\sigma_{Z^d\setminus\Lambda}} = \Phi_A \tag{2.32}$$

for all fixed $A\subset\mathbb{Z}^d$ and boundary conditions $\bar\sigma$.

Then, for any Gibbs-measure μ on the original system, corresponding to the local specification $\mu_\Lambda^{\bar\sigma_{Z^d\setminus\Lambda}}(\sigma_\Lambda)$, the renormalized measure $T\mu$ is a Gibbs-measure for the limiting interaction potential $\Phi=(\Phi_A)_{A\subset\mathbb{Z}^d}$.

The proof will be given in a moment. Assuming this result, the proof of Theorem 1 is immediate: The convergence of the renormalized potentials (2.32) is readily checked by the explicit expressions (2.4),(2.5),(2.22) with (2.17). In fact, in our case of a Kac-interaction given by a characteristic function, the potentials even become Λ-independent for Λ large enough. Uniform absolute summability at every site x is clear by the explicit estimate (2.6).

Let us finally give the

Proof of Proposition 2.2: Choose volumes $V_0\subset V_1\subset V_2$. Assuming the Gibbs-type form (2.30) we have for the conditional expectations

$$\begin{aligned}
&\mu^{\bar\sigma}_{\Lambda_2}(m_{V_0}(\sigma_{\Lambda_0}) = m_{V_0} | m_{V_1\setminus V_0}(\sigma_{\Lambda_1\setminus\Lambda_0}) = m_{V_1\setminus V_0}) \\
&= \frac{\sum_{\tilde m_{V_2\setminus V_1}} e^{-\sum_{A\subset V_2} \Phi^{\bar\sigma}_A(m_{V_0} m_{V_1\setminus V_0} \tilde m_{V_2\setminus V_1})}}{\sum_{\tilde m_{V_0}} \sum_{\tilde m_{V_2\setminus V_1}} e^{-\sum_{A\subset V_2} \Phi^{\bar\sigma}_A(\tilde m_{V_0} m_{V_1\setminus V_0} \tilde m_{V_2\setminus V_1})}} \\
&= \frac{e^{-\sum_{\substack{A\subset V_1\\ A\cap V_0\neq\emptyset}} \Phi^{\bar\sigma}_A(m_{V_0} m_{V_1\setminus V_0})} < e^{-\sum_{\substack{A\subset V_2 : A\cap V_0\neq\emptyset\\ A\cap V_2\setminus V_1\neq\emptyset}} \Phi^{\bar\sigma}_A(m_{V_0} m_{V_1\setminus V_0} \tilde m_{V_2\setminus V_1})} >_{V_2\setminus V_1}}{\sum_{\tilde m_{V_0}} e^{-\sum_{\substack{A\subset V_1\\ A\cap V_0\neq\emptyset}} \Phi^{\bar\sigma}_A(\tilde m_{V_0} m_{V_1\setminus V_0})} < e^{-\sum_{\substack{A\subset V_2 : A\cap V_0\neq\emptyset\\ A\cap V_2\setminus V_1\neq\emptyset}} \Phi^{\bar\sigma}_A(\tilde m_{V_0} m_{V_1\setminus V_0} \tilde m_{V_2\setminus V_1})} >_{V_2\setminus V_1}}
\end{aligned} \tag{2.33}$$

with the short notation

$$< f(m_{V_2\setminus V_1}) >_{V_2\setminus V_1} := \frac{\sum_{\tilde m_{V_2\setminus V_1}} f(\tilde m_{V_2\setminus V_1}) e^{-\sum_{A\subset V_2\setminus V_0} \Phi^{\bar\sigma}_A(m_{V_1\setminus V_0} \tilde m_{V_2\setminus V_1})}}{\sum_{\tilde m_{V_2\setminus V_1}} e^{-\sum_{A\subset V_2\setminus V_0} \Phi^{\bar\sigma}_A(m_{V_1\setminus V_0} \tilde m_{V_2\setminus V_1})}} \tag{2.34}$$

Now, from the summability hypothesis (2.31) follows that by choosing V_1 sufficiently large (but finite), the exponential in the brackets can be made uniformly arbitrarily close to one and thus we have

$$\begin{aligned}
&\mu^{\bar\sigma}_{\Lambda_2}(m_{V_0}(\sigma_{\Lambda_0}) = m_{V_0} | m_{V_1\setminus V_0}(\sigma_{\Lambda_1\setminus\Lambda_0}) = m_{V_1\setminus V_0}) \\
&= \frac{e^{-\sum_{\substack{A\subset V_1\\ A\cap V_0\neq\emptyset}} \Phi_A^{\bar\sigma_{\mathbb{Z}^d\setminus\Lambda_2}}(m_{V_0} m_{V_1\setminus V_0})}}{\sum_{\tilde m_{V_0}} e^{-\sum_{\substack{A\subset V_1\\ A\cap V_0\neq\emptyset}} \Phi_A^{\bar\sigma_{\mathbb{Z}^d\setminus\Lambda_2}}(\tilde m_{V_0} m_{V_1\setminus V_0})}} \times (1 + o(\Lambda_1)),
\end{aligned} \tag{2.35}$$

uniformly in $\Lambda_2 \supset \Lambda_1$.

Let us now assume that, for a given Gibbs-measure μ, the boundary condition $\bar\sigma$ is chosen s.t. $\lim_{\Lambda_2} \mu^{\bar\sigma}_{\Lambda_2} = \mu$. Taking the limit $\Lambda_2 \uparrow \mathbb{Z}^d$ we recover the renormalized measure $T\mu$ on the l.h.s. (noting that the renormalization group transformation is local!) and from the convergence of the potential to the boundary-independent expression we have the estimate

$$(T\mu)(m_{V_0} | m_{V_1\setminus V_0}) = \frac{e^{-\sum_{\substack{A\subset V_1\\ A\cap V_0\neq\emptyset}} \Phi_A(m_{V_0} m_{V_1\setminus V_0})}}{\sum_{\tilde m_{V_0}} e^{-\sum_{\substack{A\subset V_1\\ A\cap V_0\neq\emptyset}} \Phi_A(\tilde m_{V_0} m_{V_1\setminus V_0})}} \times (1 + o(\Lambda_1)) \tag{2.36}$$

Finally we can put $\Lambda_1 \uparrow \mathbb{Z}^d$. The last equation shows the continuity of the conditional expectations on the r.h.s. and their convergence to the Gibbs-formula, as desired. $\diamondsuit$

We don't need it in the paper, but let's make a simple comment on the translation-invariant case, by which we mean that both the local configuration of the original system and the map T are translation-invariant. Assume that we have the representation (2.30) and conditions (2.31) and (2.32) only for either 1) periodic boundary conditions or 2) one specific boundary condition

$\bar{\sigma}$ for which we know that the local specification of the original system converges to a particular translation-invariant Gibbs-measure μ_1. Then it follows already that $T\mu$ is Gibbsian for the same limiting potential, for all translation-invariant Gibbs measures μ.

E.g. for the case of periodic boundary conditions, this is seen as follows. Note that by compactness there is always a translation invariant weak limit point μ_1 of the corresponding finite volume measures. By the reasoning given in the proof of the proposition, $T\mu_1$ is Gibbs for the limiting potential. But from this follows that all renormalized translation-invariant are Gibbs-measures for the same potential. This is a consequence of the general dichotomy-theorem for block-transformed *translation-invariant* Gibbs-measures, by [EFS] (see [EFS] Theorem 3.4), which states that all renormalized *translation invariant* Gibbs measures are either Gibbs for the same potential or not Gibbs at all.

Appendix:
A convergence criterion for cluster-expansions for long-range graphs

Proposition A.1: *Suppose that*

$$\sum_{(P_1,\dots,P_n)_c} \prod_{i=1}^{n} \rho_{P_i} \tag{A.1}$$

is a polymer partition function, where: 'Polymers' P are graphs on the lattice $\mathbb{Z}^d$ having at least one edge. Two polymers are called compatible if they have disjoint vertex sets. The sum is over pairwise compatible families of polymers taken from a finite subset $\mathcal{P}$ of the set of graphs on $\mathbb{Z}^d$. Assume that the (possibly complex) activities ρ_P satisfy the bounds

$$|\rho_P| \le e^{-\sum_{b\in P}\tau_b} \quad \text{where} \quad \lambda := \sum_{y:y\neq x} e^{-\tau_{x,y}} \le \lambda^* \approx 0.110909 \tag{A.2}$$

for some translation invariant function $\tau_b = \tau_{x,y} \ge 0$ on the set of edges on $\mathbb{Z}^d$, where the above b-sum is over all edges of the graph P.

Then, the cluster expansion converges, i.e. the Taylor-series of the logarithm of the partition function has the representation

$$\log \sum_{(P_1,\dots,P_n)_c} \prod_{i=1}^{n} \rho_{P_i} = \sum_{\mathcal{C}} \Phi_{\mathcal{C}} \tag{A.3}$$

where the sum is over indecomposable subsets $\mathcal{C}\subset\mathcal{P}$. 'Indecomposable' means that there do not exist nonempty $\mathcal{C}_1$ and $\mathcal{C}_2$ s.t. the pairs P_1, P_2 are always compatible for $P_1 \in \mathcal{C}_1$, $P_2 \in \mathcal{C}_2$. The weight $\Phi_{\mathcal{C}} = \sum'_{I:I\in\mathbb{N}^{\mathcal{P}}} c_I \prod_{P\in\mathcal{P}} \rho_P^{I_P}$ is the sum over all monomials in the Taylor-expansion corresponding to multi-indices I with $I_P \ge 1$ for all $P \in C$ and c_I is the corresponding combinatorial factor, depending only on the incompatibility relation.

Moreover, we have the decay-estimate of the form

$$\sum_{\mathcal{C}:\mathcal{C}\ \text{icp P}} |\Phi_{\mathcal{C}}| \left(\frac{\lambda^*}{\lambda}\right)^{|\mathcal{C}|} \le a^*|P|, \quad \textit{where} \quad a^* \approx 0.633 \tag{A.4}$$

for any fixed P. Here the sum is over all clusters incompatible with P, i.e. containing at least one polymer incompatible with P and we have put $|\mathcal{C}| = \sum_{P\in\mathcal{C}} |P|$ where $|P|$ is the number of bonds of the polymer P.

Proof: The proof is based on the Kotecky-Preiss convergence-criterion [KP] for abstract polymer models plus a little combinatorics. A very nice and simple proof of the KP-criterion (with slightly weaker bounds) can be found in [BZ2] (see also [S]). It says that the hypothesis

$$\sum_{P':P'\ \text{icp P}} |\rho_{P'}| e^{a(P')+\delta(P')} \le a(P) \tag{A.5}$$

where $a(P)$ and $\delta(P)$ are weight-functions on the set of polymers, implies convergence of the cluster expansion. Furthermore it gives the estimate

$$\sum_{\mathcal{C}:\mathcal{C}\ \text{icp P}} |\Phi_{\mathcal{C}}| e^{\delta(\mathcal{C})} \le a(P) \tag{A.6}$$

for any $P \in \mathcal{P}$, where $\delta(\mathcal{C}) = \sum_{P\in\mathcal{C}} \delta(P)$.

In our present, possibly long-range case, we choose the weight-functions as $a(P) = a|P|$ and $\delta(P) = \delta|P|$ with $a, \delta > 0$ (whose values will be fixed later), and estimate

$$\sum_{P':P'\ \text{icp P}} |\rho_{P'}| e^{a(P')+\delta(P')} \le \sum_{x:x\in X(P)} \sum_{P':X(P')\ni x} |\rho_{P'}| e^{(a+\delta)|P'|} \tag{A.7}$$

where $X(P)$ is the vertex set of P. This is a certain overestimation that could be improved upon for short-range models. If we think of long range models where the number of bonds that can emanate from a vertex is large, and the nearest bonds don't have a large relative weight, the loss is very small. So we see that the hypothesis of the KP-criterion is implied if the last P'-sum is less or equal than $a/2$. By the form of the bound on the activities we assume, this is true if

$$\sum_{P:X(P)\ni x} e^{-\sum_{b:b\in P}(\tau_b - a - \delta)} \le \frac{a}{2} \tag{A.8}$$

We need an upper bound on the sum of the l.h.s. in terms of the bound on λ given our hypothesis. This is provided by the following combinatorial Lemma.

Lemma A.2: *Put $b(t) := \sum_{P:X(P)\ni x} e^{-\sum_{\{x,y\}\in P} t_{x,y}}$ where $t_{x,y}$ are translation-invariant.*

Then the bound $\sum_{y:y\neq x} e^{-t_{x,y}} \le \frac{1}{e}$ implies that $b(t) \le h^{-1}\left(\sum_{y:y\neq x} e^{-t_{x,y}}\right) - 1$, with the function $h : [1, e] \to [0, \frac{1}{e}]$, given by $h(y) := \frac{\log y}{y}$

Remark: Note that h is a strictly increasing one-to-one mapping from $[1, e]$ to $[0, \frac{1}{e}]$ and we have $x \le h^{-1}(x) - 1 = x + \ldots$ for x small. So, the Lemma shows that, for small enough weights, the sum over all polymers containing a given site, is essentially given by the sum over all single-bond polymers.

Proof: We restrict the maximal number of edges in the polymers occurring in the sum, and put $b_n(t) := \sum_{P: X(P) \ni x, 1 \le |P| \le n} e^{-\sum_{\{x,y\} \in P} t_{x,y}}$. We proceed by induction over n.

We start with the case $n = 1$. Then we have $b_1(t) = \sum_{y: y \neq x} e^{-t_{x,y}}$ which is smaller than the r.h.s. of the inequality we claim, because $x \le h^{-1}(x) - 1$.

Next we assume that the desired equality holds for b_n. We want to show that it holds for b_{n+1}. Now, every polymer containing the site x with at most $n+1$ bonds can be decomposed (in a possibly non-unique way) into a bond emerging from x going to some other site y and a polymer containing the site y with at most n bonds. So, we have the inequality

$$
\begin{aligned}
b_{n+1}(t) \le & \sum_{y: y \neq x} e^{-t_{x,y}} \left(1 + \sum_{P: X(P) \ni y, 1 \le |P| \le n} e^{-\sum_{b \in P} t_b} \right) \\
& + \frac{1}{2!} \sum_{\substack{y_1: \\ y_1 \neq x}} \sum_{\substack{y_2: \\ y_2 \neq x, y_1}} e^{-t_{x,y_1}} e^{-t_{x,y_2}} \left(1 + \sum_{P: X(P) \ni y_1, 1 \le |P| \le n-1} e^{-\sum_{b \in P} t_b} \right) \\
& \quad \times \left(1 + \sum_{P: X(P) \ni y_2, 1 \le |P| \le n-1} e^{-\sum_{b \in P} t_b} \right) \\
& + \ldots + \\
& + \frac{1}{k!} \sum_{y_1: y_1 \neq x} \cdots \sum_{y_k: y_k \neq x} e^{-t_{x,y_1}} \ldots e^{-t_{x,y_k}} \prod_{i=1}^{k} \left(1 + \sum_{P: X(P) \ni y_1, 1 \le |P| \le n-k} e^{-\sum_{b \in P} t_b} \right) + \ldots
\end{aligned}
\tag{A.9}
$$

Bounding the r.h.s. by an exponential gives

$$
\begin{aligned}
b_{n+1}(t) & \le \sum_{k=1}^{n} \frac{1}{k!} \left(\sum_{y: y \neq x} e^{-t_{x,y}} \left(1 + \sum_{P: X(P) \ni x, 1 \le |P| \le x} e^{-\sum_{b \in P} t_b} \right) \right)^k \\
& \le \exp \left(\sum_{y: y \neq x} e^{-t_{x,y}} (1 + b_n(t)) \right) - 1,
\end{aligned}
\tag{A.10}
$$

by translation-invariance. Using the induction hypothesis on b_n we have from this

$$
b_{n+1}(t) \le \exp \left(\sum_{y: y \neq x} e^{-t_{x,y}} \, h^{-1} \left(\sum_{y: y \neq x} e^{-t_{x,y}} \right) \right) - 1 = h^{-1} \left(\sum_{y: y \neq x} e^{-t_{x,y}} \right) - 1 \tag{A.11}
$$

due to the property of the function h^{-1}. This concludes the proof of Lemma (A.2). $\diamondsuit$

From the lemma we have

$$\sum_{P:X(P)\ni x} e^{-\sum_{b:b\in P}(t_b-a-\delta)} \leq h^{-1}\left(\sum_{y:y\neq x} e^{-(\tau_{x,y}-a-\delta)}\right)-1 \tag{A.12}$$

So, (A.8) is implied if the r.h.s. is less or equal than $\frac{a}{2}$. This is equivalent to

$$e^{\delta}\lambda \leq e^{-a}h(1+\frac{a}{2}) = e^{-a}\frac{\log(1+\frac{a}{2})}{1+\frac{a}{2}} \tag{A.13}$$

But maximizing numerically the r.h.s. of this inequality over a gives the value λ^* with the maximizer a^* with values given in (A.2) and (A.4).

So, for $\lambda \leq \lambda^*$, we really get convergence (A.3) from (A.13) by the abstract KP-criterion. We get decay (A.4) with the best constant by putting $e^{\delta} := \frac{\lambda^*}{\lambda}$. This concludes the proof of Proposition A.1.$\diamondsuit$

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